

Asset Prices and Wealth Inequality*

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Abstract

Wealthy households disproportionately invest in equity, causing equity returns to generate fluctuations in wealth inequality. To examine the macro effect of these movements in the wealth distribution, I build a model in which agents have heterogeneous exposures to aggregate shocks. I show that the tail index of the wealth distribution depends of the average logarithmic return of top households. The model generates a two-way feedback between wealth inequality and asset prices, which magnifies the response of wealth inequality to aggregate shocks in the short-run but reduces it in the longer-run. The model, calibrated on U.S. data, can account for a large fraction of the fluctuations of asset prices and wealth inequality over the 20th century.

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1 Introduction

Recent empirical studies have documented important fluctuations in wealth inequality over the past century.¹ These fluctuations give rise to two important questions: what drives the dynamics of wealth inequality over time? And what are the effect of these fluctuations on aggregate quantities? In this paper, I address these questions by studying the interplay between asset prices and wealth inequality.

I focus on the following mechanism. Agents at the top of the wealth distribution are more exposed to aggregate risk. As a consequence, after a positive aggregate shock, investors at the top of the wealth distribution gain more than the rest, i.e. wealth inequality increases. As wealth is redistributed towards wealthy agents, the aggregate demand for assets increases, which, in turn, increasing asset valuations. The equilibrium adjustment of asset valuations amplifies the effect of the aggregate shock on wealth inequality in the short-run but dampens it in the long-run. I show that this mechanism can account for a large fraction of the dynamics of asset prices and top wealth inequality over the 20th century.

The paper proceeds in three steps. I first document that wealthy households are twice as exposed to equity returns relative to the rest of the population, using data from [Kopczuk and Saez \(2004\)](#) and *Forbes*. More precisely, in response to a realized stock return of 10%, the average wealth in the economy increases by 4.6% while the average wealth in the top 0.01% increases by 9.2%. Because stock market returns are volatile and i.i.d over time, I show that this mechanism can explain a large fraction of the fluctuations in top wealth shares over the 20th century.

Motivated by this reduced form evidence, I build a model in which agents have heterogeneous exposures to aggregate shocks. In the model, a subset of the population (“entrepreneurs”) must hold a concentrated position in their firms, while the rest (“households”) can freely trade equity. I characterize the dynamics of the wealth distribution implied by the model. Despite the presence of aggregate shocks, I am able to obtain an analytical characterization for the tail index of the wealth distribution: it depends on the *average logarithmic* return of top households relative to the growth of the economy.

Third, I calibrate the model using U.S. data. In particular, I use my measured elasticity of top wealth shares to infer the equity exposure of entrepreneurs, and I use the tail index of the wealth distribution to infer their consumption rate. The calibrated model features a strong interplay between asset prices and wealth inequality: after a positive aggregate shock, entrepreneurs earn more than the rest, which generates an excess demand for assets, leading to higher asset prices in equilibrium. Overall, the model can account for an important fraction of the dynamics of top wealth inequality and asset prices observed in the U.S.. Using continuous-time methods, I characterize the impulse-response of individual wealth following an aggregate shock. I find that the equilibrium adjustment of asset prices amplifies the short-run effect of aggregate shocks on wealth

¹See, for instance, [Wolff \(2002\)](#), [Kopczuk and Saez \(2004\)](#), and [Saez and Zucman \(2016\)](#).

inequality while dampening their long run impact.

Literature review. This paper contributes to a large literature examining the heterogeneity in equity holdings across the distribution of households (Guiso et al., 1996; Carroll, 2000; Campbell, 2006; Wachter and Yogo, 2010; Roussanov, 2010; Bach et al., 2015; Kacperczyk et al., 2018). In particular, Parker and Vissing-Jørgensen (2010) document that the income of top percentiles has gradually become more exposed to aggregate shocks. Mankiw and Zeldes (1991) and Malloy et al. (2009) document that the consumption of rich stockholders is more exposed to stock market returns. In contemporaneous work, Kuhn et al. (2020) measures the elasticity of the top 10% wealth share to stock market returns using data from the Survey of Consumer Finances (SCF). Relative to this paper, I use data from Kopczuk and Saez (2004) and Forbes to focus on the top of the wealth distribution. Overall, I find that households at the top of the wealth distribution are twice as exposed to equity returns relative to the average household in the population, which is much larger than existing estimates.²

The paper also contributes to the theoretical literature on wealth inequality. Random growth theories of the wealth distribution include Jones (2015), Piketty and Zucman (2015), Gabaix et al. (2016), and Cao and Luo (2016). More recently, Luttmer (2012) and Gabaix et al. (2016) characterize the dynamics of the wealth distribution between two steady states. This paper improves on the literature by characterizing the wealth distribution in a stochastic economy. Despite the presence of aggregate shocks, I obtain a simple characterization of the tail index of the wealth distribution in terms of the *average logarithmic* wealth growth of top households relative to the economy. This new theoretical result connects the paper to Kelly (1956), Blume et al. (1992), and Borovička (Forthcoming), who stress the importance of this quantity for long-run survival in infinite-horizon economies with aggregate shocks.

This paper also contributes to a more empirical literature documenting the dynamics of wealth inequality over time in the U.S. (Saez and Zucman, 2016, Smith et al., 2021). While a number of studies focus on the role of the risk-free rate of return in shaping the wealth distribution (Piketty (2014), Acemoglu and Robinson (2015)), I document a more important role for the rate of return on risky assets (Wolff and Marley (1989), Kuhn et al. (2020)). For simplicity, my model abstracts from other sources of wealth inequality dynamics discussed in the literature, such as taxes (Hubmer et al., 2021), saving rates (Mian et al., 2020), idiosyncratic volatility (Gomez, 2018; Benhabib et al., 2019; Atkeson and Irie (2020)), or technology (Moll et al., 2021).

This paper also contributes to the large asset pricing literature with heterogeneous agents (Dumas, 1989; Guvenen 2009; Chan and Kogan, 2002; Basak and Cuoco, 1998; Gomes and Michaelides, 2008; Brunnermeier and Sannikov, 2014; He and Krishnamurthy, 2012, and, in particular, Gârleanu and Panageas, 2015, who study asset prices in an overlapping generation model with two types of

²In particular, it is an order of magnitude larger than Kuhn et al. (2020)'s measure of the elasticity of the top 10% wealth share to stock market returns. One reason is that I focus on top percentiles. Another reason is that I measure the effect of stock market returns over longer horizons to account for measurement error.

agents). This theoretical literature seldom considers its implication on wealth inequality. An open question in the literature is: does the wealth distribution move enough over the business cycle to account for the excess volatility of asset prices in equilibrium?³ My paper addresses this question by calibrating a model on the joint dynamics of asset prices and wealth inequality. In particular, I show that using moments about the wealth distribution, such as the volatility of top wealth shares and the tail index of the wealth distribution, is key to discipline the degree of heterogeneity across households.

Finally, this paper is connected to a growing literature examining the interplay between aggregate quantities and wealth inequality. The work of [Gollier \(2001\)](#) is an early example that examines theoretically the importance of the wealth distribution for asset prices. [Toda and Walsh \(2016\)](#) document that fluctuations in income inequality negatively predict future excess stock returns—I confirm this result by looking at the relationship between top wealth inequality and predicting future excess stock returns. [Barczyk and Kredler \(2016\)](#) examine theoretically the role of inequality and incomplete markets on asset prices. [Eisfeldt et al. \(2016\)](#) discuss the joint relation between the wealth distribution and asset prices across markets with different expertise. [Favilukis \(2013\)](#) examines the role of changes in participation cost and wage inequality on asset prices. More recently, [Auclert and Rognlie \(2017\)](#) and [Straub \(2019\)](#) study the effect of a secular rise in income inequality on interest rates — in comparison, I focus on the business cycle fluctuations in wealth inequality and asset prices in response to aggregate shocks.

Roadmap. The rest of the paper is organized as follows. In [Section 2](#), I measure and document the equity exposure at the top of the wealth distribution. In [Section 3](#), I examine a model of the wealth distribution in which agents have heterogeneous exposures to aggregate shocks and I characterize the implied wealth distribution. In [Section 4](#), I calibrate the model using U.S. data. [Section 5](#) concludes.

2 Equity Exposure in the Right Tail

In this section, I quantify the effect of stock market returns on the dynamics of top wealth shares. [Section 2.1](#) presents the data, [Section 2.2](#) measures the response of wealth inequality to stock market returns, while [Section 2.3](#) discusses the cumulative effect of stock market returns for the fluctuations in wealth inequality over the 20th century.

³Reflecting this sentiment, [Cochrane \(2017\)](#) writes that “[the heterogeneous agents] model faces challenges and opportunities in the micro data just as the idiosyncratic risk model does. Do the ‘high-beta rich’ really lose so much in bad times? Can the model quantitatively account for return predictability? But that investigation has not really started.”

2.1 Data

I am interested in measuring changes in the wealth distribution and their relationship to stock returns. Therefore, I need yearly estimates of the wealth distribution that cover several business cycles. I use two datasets that, together, cover most of the last 100 years. The first dataset is the annual series of top wealth shares constructed by [Kopczuk and Saez \(2004\)](#). This series is constructed from estate tax returns, which report the wealth of deceased households. [Kopczuk and Saez \(2004\)](#) estimate the living's wealth distribution from the deceased's wealth distribution using the mortality multiplier technique, which amounts to weighting each estate tax return by the inverse probability of death (depending on age and gender). The series is constructed using the whole universe of estate tax returns during the 1916–1945 period and a stratified sample of micro files for 1965, 1969, 1972, 1975, and 1982–2000.

I supplement the series of top wealth shares from estate tax returns with the list of the wealthiest 400 Americans constructed by *Forbes* every year since 1982, which offers an unparalleled view on the right tail of the wealth distribution. The list is created by a dedicated staff of the magazine, based on a mix of public and private information.⁴

I focus on the percentile group that includes the entirety of households in the Forbes 400 list in 2017. Because a percentile includes a constant fraction of the total population, this top percentile only includes 264 households in 1983.

I measure average household wealth in the U.S from the U.S. Financial Accounts after 1945. Before 1945, I use data from [Wolff and Marley \(1989\)](#), who report estimates based on the same concepts and methods as the U.S. Financial Accounts. Finally, I use yearly stock market returns and risk-free rates from [Shiller \(2015\)](#).⁵

2.2 Empirical Findings

Specification. I define the wealth exposure of households in a given top percentile group as the slope estimate in a regression of the logarithmic growth of the average wealth in the group on excess logarithmic stock market returns; that is,

$$\log \left(\frac{W_{t+h}^G}{W_t^G} \right) - h \log R_{f,t+1} = \alpha_{Gh} + \beta_{Gh} (\log R_{M,t+1} - \log R_{f,t+1}) + \epsilon_{Gh,t+1}, \quad (1)$$

where W_t^G denotes the average wealth of households in group G in year t , $\log R_{M,t}$ denotes the log stock market return, and $\log R_{f,t}$ denotes the log risk-free rate.

⁴*Forbes* reports that “we pored over hundreds of Securities Exchange Commission documents, court records, probate records, federal financial disclosures and Web and print stories. We took into account all assets: stakes in public and private companies, real estate, art, yachts, planes, ranches, vineyards, jewelry, car collections and more. We also factored in debt. Of course, we don’t pretend to know what is listed on each billionaire’s private balance sheet, although some candidates do provide paperwork to that effect.”

⁵Available at <http://aida.wss.yale.edu/~shiller/data/chapt26.xlsx>.

I examine the effect of a stock market return on wealth growth over an horizon h , which may be larger than one year. The reason is that a large share of wealth in top percentiles is held in privately held assets, which are not typically traded. Since the valuation of this non-traded wealth reacts sluggishly to changes in the stock market, this stale pricing may lead an econometrician to underestimate the wealth exposure of households over small horizons.^{6,7}

Results. Figure 1 plots the estimates of β_{Gh} obtained by making the horizon h vary from $h = 1$ to $h = 8$ in equation (1) for four group of households: all households, households in the top 1%–0.1%, households in the top 0.1%–0.01%, and households in the top 0.01%. Note that this can be interpreted as the impulse response of the average wealth in each group to a stock market return shock (as in Jordà, 2005). The estimates gradually increase up to $h = 4$, consistent with the idea that the valuation of household wealth lags public indices. The key finding is that, at each horizon, the estimated exposure β_{Gh} increases monotonically with the top percentile.

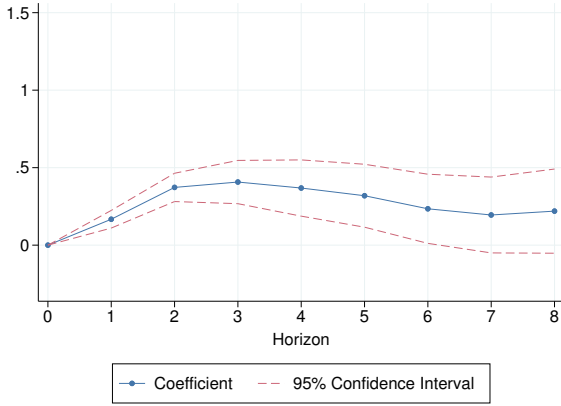
Table 1 reports the estimates for β_{G4} , the wealth exposure to the stock market at the four-year horizon. The estimated exposure β_{G4} increases monotonically with the top percentiles, from $\beta = 0.46$ for the average household to $\beta = 0.92$ for households in the top 0.01%; in other words, households at the top of the wealth distribution are twice as exposed to stock market returns relative to the average household in the economy. The 5th column of Panel A reports the wealth exposure for households in the Forbes Top 400. The estimate is very similar in magnitude to the estimate for the households in the top 0.01% from tax data. This is reassuring because these datasets are constructed from two completely different methodologies and time periods.

Composition effects. Top percentiles do not necessary include the same individuals over time—some people enter and drop from the top every year. In particular, one concern is that there may be more entrants in top percentiles following high stock market returns. This effect would tend to increase the response of the average wealth in the top wealth shares to stock market returns relative to the average wealth response of households in the top. To address this issue, following Gomez (2018), I leverage the panel dimension of Forbes 400 to construct a series of the average wealth growth of households *initially* in the top percentile (whether or not they remain in the top in the subsequent period). This “within” yearly series differs from the total growth of the top 400 by removing the effect of compositional changes on the growth of top wealth shares.

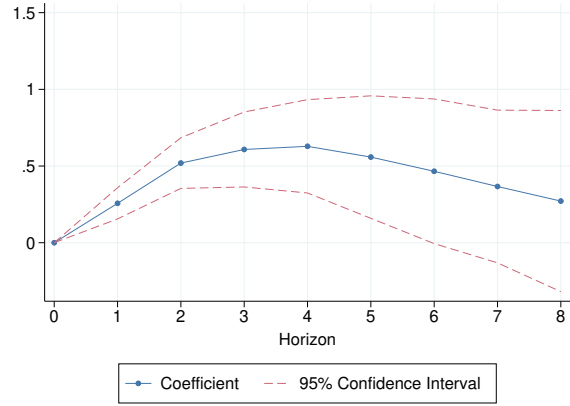
As shown in Figure 1 and Table 1, I obtain very similar estimates for β_{hG} using this alternative series. This means that compositional changes play little role in driving the reaction of top wealth shares to the stock market exposure of the Forbes 400. One reason is that changes in idiosyncratic

⁶The valuation of this private wealth depends on the source. For estate tax, an external appraiser does the valuation of non-tradable assets. *Forbes* uses the valuation implied by the most recent financing round or the prevailing price-to-earnings ratios for similar public companies.

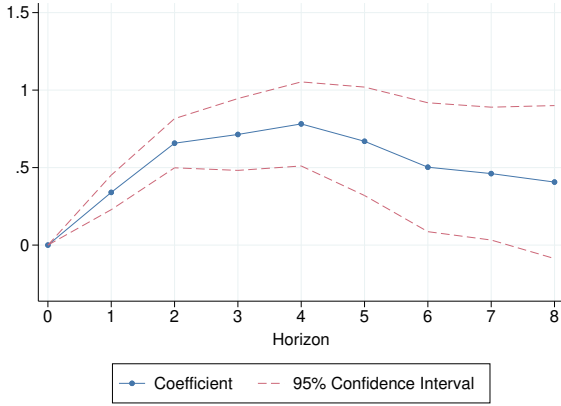
⁷Relatedly, Brav et al. (2002) and Malloy et al. (2009) show that reported consumption growth for richer households is more correlated to aggregate consumption growth at a longer horizon. In a different context, Getmansky et al. (2004) argue that this stale pricing may allow hedge funds to artificially decrease their exposure to stock market returns.



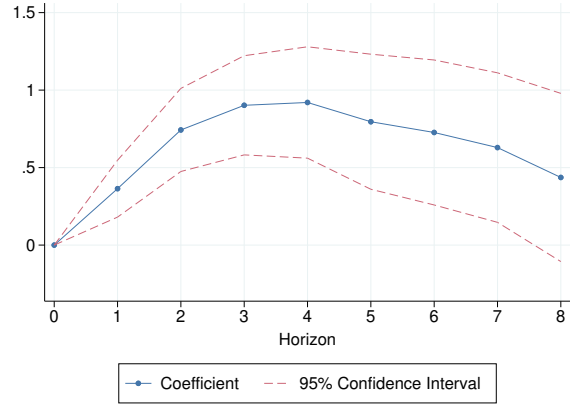
(a) Household Wealth



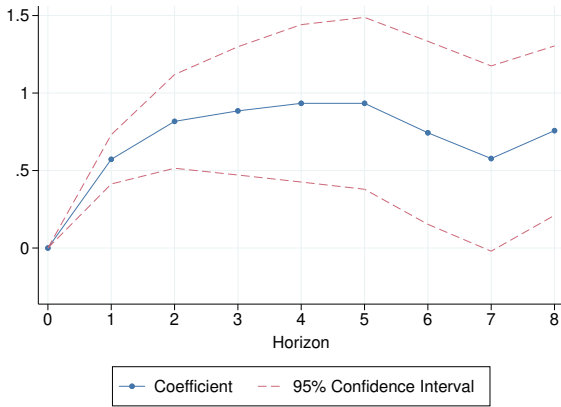
(b) Top 1%–0.1%



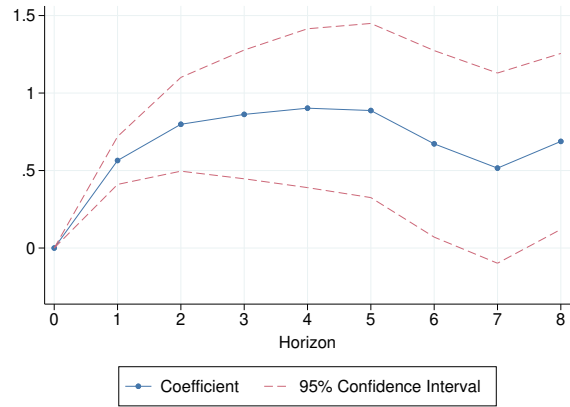
(c) Top 0.1%–0.01%



(d) Top 0.01%



(e) Forbes 400



(f) Forbes 400 (within)

Figure 1: Exposure β of Top Wealth Group to Stock Market Returns at Different Horizons

Notes: This figure reports the estimates for β_{Gh} estimated via the regression (1) from $h = 0$ to $h = 8$ as well as the 5%–95% confidence intervals using Newey-West with four lags. Each figure corresponds to a different group of households. Figure 1a corresponds to all U.S households. Figures 1b-1d correspond to increasing top percentiles in the wealth distribution using data from [Kopczuk and Saez \(2004\)](#). Figure 1e corresponds to Forbes 400 while Figure 1f corresponds to its cumulative “within” term.

Table 1: Exposure to Stock Returns across the Wealth Distribution

	Growth of Average Wealth					
	Estate Tax Returns				Forbes 400	
	U.S.	1%–0.1%	0.1%–0.01%	Top 0.01%	All	Within
	(1)	(2)	(3)	(4)	(5)	(6)
<i>Panel A: Wealth</i>						
Excess Stock Returns	0.46*** (0.13)	0.63*** (0.16)	0.78*** (0.14)	0.92*** (0.18)	0.93*** (0.25)	0.90*** (0.25)
R^2	0.19	0.24	0.38	0.34	0.31	0.29
Period	1917-1997	1917-1997	1917-1997	1917-1997	1983-2013	1983-2013
N	53	53	53	53	31	31
<i>Panel B: Wealth Share</i>						
Excess Stock Returns		0.17*** (0.05)	0.32*** (0.06)	0.46*** (0.12)	0.55*** (0.19)	0.51*** (0.18)
R^2		0.20	0.38	0.23	0.21	0.21
Period		1917-1997	1917-1997	1917-1997	1983-2013	1983-2013
N		53	53	53	31	31

Notes: This table reports the results of the regression of the excess wealth growth of households in a given percentile group on the excess stock returns, i.e., equation (1) with $h = 4$. The dependent variables are the growth of wealth in Panel A and the growth of wealth shares in Panel B. Each column corresponds to a different group of households. The first column corresponds to all U.S. households, and Columns (2)–(4) corresponds to increasing top percentiles in the wealth distribution, using data from [Kopczuk and Saez \(2004\)](#). Column (5) corresponds to the top 3% of the top 0.01%. This last percentile is chosen so that the group includes the 400 wealthiest individuals in 2015. Estimation is done via OLS. Standard errors are in parentheses and are estimated using Newey-West with four lags. *, **, *** indicate significance at the 10%, 5%, 1% levels, respectively.

volatility are not very correlated with stock market returns. Another reason is that, as discussed in [Gomez \(2018\)](#), typical fluctuations in idiosyncratic volatility have a much smaller effect on top wealth shares compared to fluctuations in stock market returns.

Elasticity of top wealth shares. Since top percentiles are more exposed to the stock market compared to the average household in the economy, high stock market returns increase top wealth inequality. Table 1 (Panel B) confirms this relationship by regressing top wealth *shares* on stock market returns. The estimate 0.46 for the top 0.01% corresponds to the difference of exposure between households at the top and the average household ($\approx 0.92 - 0.46$).

Robustness. I conduct a few robustness checks in the appendix. First, in Appendix B.1, I show that the rise in wealth exposure at the top of the wealth distribution is consistent with the rise in the share of their portfolio held in equity as measured in the Survey of Consumer Finances.

Second, in Appendix B.2, I show that Fama-French factors or bond factors do not seem to play a big role in driving the dynamics of wealth in top percentiles, which is why I focus on stock market returns as the main drivers for the dynamics of the average wealth in top percentiles in the rest of the paper.

Third, in Appendix B.3, I reproduce my results using the alternative series on top wealth shares

constructed in [Saez and Zucman \(2016\)](#) (updated in [Piketty et al., 2018](#)). The series is constructed from income tax returns using a capitalization method. However, due to data limitations, the series builds in smoothing over time (i.e., using posterior data to construct top shares in a given year), which tends to smooth out business cycle dynamics of top wealth inequality. Still, I find that my main empirical results hold using this alternative series.

2.3 Cumulative effect

Because stock market returns are very volatile and approximately i.i.d., and because top wealth shares tend to be very persistent, the over-exposure of top households to stock market returns can potentially generate large fluctuations in top wealth inequality over time. To visualize this effect, I construct a synthetic version of top wealth shares which is only driven by past stock market returns.

Formally, I compare the cumulative logarithmic growth of the share of wealth owned by the top 0.01% over the 20th century to a synthetic series X_t constructed recursively as $X_{1916} = 0$ and for $t \geq 1916$:

$$X_{t+1} = \alpha + \rho X_t + \beta (\log R_{Mt+1} - \log R_{ft+1}), \quad (2)$$

where $\beta = 0.45$ corresponds to the elasticity of top wealth shares to stock market returns measured in [Table 1](#) and ρ corresponds to the autocorrelation of the logarithmic top wealth share in sample. As reported in [Appendix Table A5](#), I measure 0.96 in the data with a 90% confidence interval of 0.89 and 1.02. I therefore plot the synthetic series with $\rho = 0.9$ and $\rho = 1$, which gives similar results.⁸ The intercept α is chosen so that $X_T = 0$, since the overall logarithmic growth of the top 0.01% wealth share during the time period is approximately zero in the data.⁹

[Figure 2](#) compares this synthetic version of the top 0.01% wealth share to the actual series from [Kopczuk and Saez \(2004\)](#) and the Forbes 400 (as well as the top 0.01% series from [Saez and Zucman \(2016\)](#)). The figure shows that the synthetic series can explain a substantial fraction of the dynamics of the top 0.01% wealth share over the 20th century. In particular, stock market returns explain the persistent decline in wealth inequality during the Great Depression, its rise immediately after WW2, as well as the sharp rise in wealth inequality during the dot-com bubble.

Note, however, that the synthetic series cannot fully account for all the movements in wealth inequality over the 20th century. In particular, excess stock market returns cannot fully explain the decline in inequality in the 1940s or the rise in inequality beginning in the 1980s. While I abstract from these additional dynamics in the rest of the paper, the existing literature on top wealth inequality suggests some potential causes beyond stock market returns, such as changes

⁸Note that the difference between the two is small because the available sample is only 100 years. It would grow larger over a bigger time sample.

⁹That is, $\alpha = -\beta \left(\sum_{t=0}^T \rho^{T-t} (\log R_{Mt} - \log R_{ft}) \right) / \sum_{t=0}^T \rho^{T-t}$.

in taxes (Hubmer et al., 2021), saving rates (Mian et al., 2020), idiosyncratic volatility (Gomez, 2018; Benhabib et al., 2019; Atkeson and Irie (2020)), or technology (Moll et al., 2021).

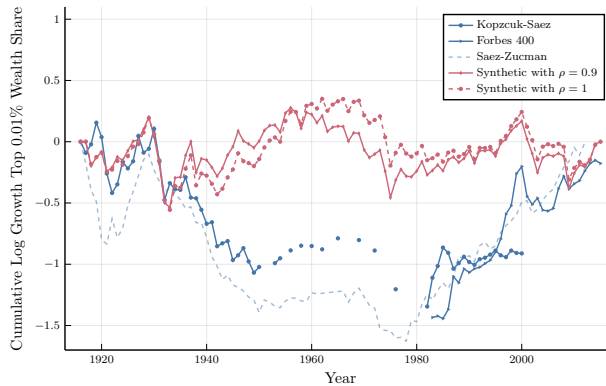


Figure 2: Cumulative Effect of Stock Market Returns on Top 0.01% Wealth Share

Notes: This figure reports the cumulative logarithmic growth of the share of wealth owned by the top 0.01% according to Kopczuk and Saez (2004), Forbes 400, Saez and Zucman (2016), as well as the synthetic series implied by cumulative lagged returns (2). The Top 0.01% wealth share according to Forbes 400 is constructed using a Pareto interpolation as in Vermeulen (2018) (i.e., as $\log(\text{Wealth Share Forbes 400}) - (1 - 1/\zeta) \times \log(0.03)$ with $\zeta = 1.5$, which reflects the fact that the population in Forbes 400 accounts for 3% of the top 0.01%.

3 Model

Motivated by the reduced-form evidence presented in the previous section, I build an asset pricing model in which agents have heterogeneous exposure to aggregate risk. Section 3.1 presents the model, Section 3.2 solves for the Markovian equilibrium, and Section 3.3 characterizes the wealth distribution implied by the model.

3.1 Setup

The model is a continuous time, pure-exchange economy with two types of agents: “households,” who can freely trade firms, and “entrepreneurs,” who must be disproportionately exposed to the firm they are born with.

Demographics. Demographics follows the perpetual youth model of Blanchard (1985): agents face a constant hazard rate of death δ and population size grows at rate η . This implies that during a short period of time dt , a proportion δdt of the population dies while a proportion $(\delta + \eta) dt$ is born. This renewal of the population over time is key for the wealth distribution to be stationary.

A proportion π of agents are born as “entrepreneurs” while the rest are born as “households”. Denote \mathbb{I}_{Ht} the set of households, \mathbb{I}_{Et} the set of entrepreneurs, and $\mathbb{I}_t \equiv \mathbb{I}_{Ht} \cup \mathbb{I}_{Et}$ the set of all agents in the economy at time t . I assume that the wealth of agents who die during a period dt is

uniformly redistributed to agents who are born during dt .¹⁰

Technology. Aggregate income per capita Y_t follows a geometric random walk; that is,

$$\frac{dY_t}{Y_t} = g dt + \sigma dZ_t,$$

where $(Z_t)_{t \in \mathbb{R}}$ is a standard Brownian motion that corresponds to aggregate shocks.

Each agent is born with a tree that delivers a stochastic income which depreciates over time. Formally, a tree i produces an income $Y_{it} = \vartheta_{it} Y_t$, where ϑ_{it} evolves as

$$\frac{d\vartheta_{it}}{\vartheta_{it}} = -\phi dt + \nu dB_{it},$$

where ν is the idiosyncratic volatility of the income process and $(B_{it})_{t \in \mathbb{R}}$ is a standard Brownian motion that corresponds to shocks specific to the tree i . The value of ϑ_{it} at birth is set to $(\eta + \phi)/(\eta + \delta)$ so that aggregating the output of all trees in existence sums up to the aggregate endowment Y_t .¹¹

Market. Agents in the economy can trade risk-free claims in zero net supply as well as claims to trees. Denote r_t the risk-free rate and p_t the market value of a tree relative to its output, which evolves according to

$$\frac{dp_t}{p_t} = \mu_{pt} dt + \sigma_{pt} dZ_t,$$

where μ_{pt} and σ_{pt} will be determined in equilibrium. The instantaneous return of holding one tree is the sum of the income yield and the growth of the tree's price:

$$\frac{dR_{it}}{R_{it}} \equiv \frac{1}{p_t} dt + \frac{d(Y_{it} p_t)}{Y_{it} p_t}.$$

¹⁰Following Blanchard (1985) and Gârleanu and Panageas (2015), a previous version of the paper assumed that the wealth of agents who die during dt was redistributed to all existing agents through an annuity market. However, this assumption implies that the return earned by existing fortunes increases by δ every period, which leads to counterfactual implications on the wealth.

¹¹Indeed, aggregating the output of trees with respect to their age gives

$$\int_{s=0}^{\infty} (\eta + \delta) e^{-\eta s} \left(\frac{\eta + \phi}{\eta + \delta} e^{-\phi s} Y_t \right) ds = Y_t.$$

Applying Ito's lemma, the instantaneous return of holding a tree can be rewritten as

$$\begin{aligned}\frac{dR_{it}}{R_{it}} &= \frac{1}{p_t} dt + \frac{dY_{it}}{Y_{it}} + \frac{dp_t}{p_t} + \frac{dY_{it}}{Y_{it}} \frac{dp_t}{p_t} \\ &= \underbrace{\left(\frac{1}{p_t} + g - \phi + \mu_{pt} + \sigma\sigma_{pt} \right)}_{\equiv \mu_{Rt}} dt + \underbrace{(\sigma + \sigma_{pt})}_{\equiv \sigma_{Rt}} dZ_t + \nu dB_{it}.\end{aligned}\quad (3)$$

Households. Households have [Duffie and Epstein \(1992\)](#) recursive preferences.¹² More precisely, a household i with a consumption process $\{C_{it}\}$ has a utility defined recursively by

$$\begin{aligned}V_{it} &= E_t \left[\int_t^\infty f(C_{iu}, V_{iu}) du \right], \\ \text{with } f(C, V) &= \rho \frac{1 - \gamma}{1 - 1/\psi} V \left(\frac{C^{1-1/\psi}}{((1 - \gamma)V)^{\frac{1-1/\psi}{1-\gamma}}} - 1 \right).\end{aligned}$$

These preferences are characterized by three parameters: the subjective discount rate (SDR) ρ , the elasticity of intertemporal substitution (EIS) ψ , and the coefficient of relative risk aversion (RRA) γ . As discussed in [Gârleanu and Panageas \(2015\)](#), in the context of a model in which agents die at rate δ , the SDR ρ should be seen as the sum of a "true" impatience rate ($\hat{\rho}$) and the hazard rate of death (δ).

Households can freely sell their initial tree and invest in a diversified portfolio of trees. Formally, household $i \in \mathbb{I}_{Ht}$ chooses a share of wealth invested in trees, α_{it} , and a consumption rate $c_{it} = C_{it}/W_{it}$ to maximize their utility. The Hamilton-Jacobi-Bellman (HJB) equation corresponding to this problem is

$$\begin{aligned}0 &= \max_{\alpha_{it}, c_{it}} \left\{ f(c_{it} W_{it}, V_{it}) dt + E_t[dV_{it}] \right\} \\ \text{with } \frac{dW_{it}}{W_{it}} &= (r_t + \alpha_{it}(\mu_{Rt} - r_t) - c_{it}) dt + \alpha_{it} \sigma_{Rt} dZ_t.\end{aligned}\quad (4)$$

Entrepreneurs. In contrast with households, entrepreneurs are disproportionately exposed to equity. Formally, I assume that entrepreneurs must hold an exogenous share of wealth α_{Et} in the tree they are born with:

$$\alpha_{Et} = \min \left(\alpha_E, \frac{\int_{i \in \mathbb{I}_t} W_{it} di}{\int_{i \in \mathbb{I}_{Et}} W_{it} di} \right).\quad (5)$$

The upper bound on the risk exposure α_{Et} ensures that entrepreneurs are not required to own more trees than there exists in the economy. This constraint will be very rarely binding in equilibrium, but it is necessary to solve the model globally.

¹²They are a continuous time version of the recursive preferences of [Epstein and Zin \(1989\)](#).

For simplicity, I take this equity constraint as exogenous and I remain agnostic about its origin. As in Brunnermeier and Sannikov, 2014; Di Tella, 2016; Haddad, 2012, this constraint could come from a moral hazard or asymmetric information problems. Alternatively, the over-exposure of entrepreneurs could represent optimism in their projects (Moskowitz and Vissing-Jørgensen, 2002), a preference for idiosyncratic volatility (Roussanov, 2010), or a higher risk tolerance (Gârleanu and Panageas, 2015, Silva, 2016).¹³

To restrict the number of free parameters, I assume that entrepreneurs have Epstein-Zin utility with an EIS of one (i.e., log utility). This number corresponds to the estimates for Vissing-Jørgensen (2002) for the elasticity stockholders at the top of the wealth distribution. This choice also simplifies the model because it implies that the consumption rate of entrepreneurs, denoted c_{Et} , is constant over time: $c_{Et} = \rho_E$, where ρ_E denotes their SDR.

With these assumptions, the wealth of an entrepreneur $i \in \mathbb{I}_{Et}$ evolves as

$$\frac{dW_{it}}{W_{it}} = (r_t + \alpha_{Et}(\mu_{Rt} - r_t) - \rho_E) dt + \alpha_{Et}\sigma_{Rt} dZ_t + \alpha_{Et}\nu dB_{it}. \quad (6)$$

Note that this law of motion for wealth is a direct function of the entrepreneurs' fixed equity share α_{Et} and of their fixed consumption rate $c_{Et} = \rho$. This will make it easier to calibrate the model based on the observed wealth dynamics of agents at the top of the wealth distribution.

Finally, note that the wealth of entrepreneurs in (6) is exposed to the idiosyncratic risk of the tree they are born with. While this will not affect the aggregate demand for goods and assets in equilibrium (as entrepreneurs have a fixed equity share and consumption rate), this is important to generate a realistic wealth distribution.

Equilibrium. I now define the notion of an equilibrium in this model. An equilibrium is a asset of price processes $(p_t)_{t \in \mathbb{R}}$, $(r_t)_{t \in \mathbb{R}}$ and decision processes $(c_{Ht})_{t \in \mathbb{R}}$, $(\alpha_{Ht})_{t \in \mathbb{R}}$ such that

1. Given the price processes, decisions solve the consumption-savings problems of the household
2. The market for goods and risky assets clear; that is

$$\int_{i \in \mathbb{I}_t} c_{it} W_{it} di = Y_t |\mathbb{I}_t| \quad (7)$$

$$\int_{i \in \mathbb{I}_t} \alpha_{it} W_{it} di = p_t Y_t |\mathbb{I}_t|. \quad (8)$$

By Walras's law, the market for risk-free claims clears automatically.

¹³A previous version of this paper modeled "entrepreneurs" as individuals with a higher risk tolerance, rather than individuals with a constraint on portfolio holdings — this alternative modelling choices leads to similar quantitative results.

3.2 Solving the Model

I now outline the main steps in deriving the solution in this section. See the Appendix C.1 for a detailed derivation.

Household optimal policy. Given homothetic preferences and linear budget constraints, we know that the value function of households takes the form:

$$V_{it} = \frac{(\chi_t W_{it})^{1-\gamma}}{1-\gamma}, \quad (9)$$

where the process χ_t captures the investment opportunities the faced by the households. This process follows a diffusion process:

$$\frac{d\chi_t}{\chi_t} = \mu_{\chi t} dt + \sigma_{\chi t} dZ_t,$$

where $\mu_{\chi t}$ and $\sigma_{\chi t}$ will be determined in equilibrium.

Plugging (9) into the household's HJB (4) and applying Ito's lemma gives

$$0 = \max_{c_{it}, \alpha_{it}} \left\{ \frac{\rho}{1-1/\psi} \left(\left(\frac{c_{it}}{\chi_t} \right)^{1-1/\psi} - 1 \right) + r_t + \alpha_{it}(\mu_{Rt} - r_t) - c_{it} + \mu_{\chi t} - \frac{\gamma}{2} \left(\alpha_{it}^2 \sigma_{Rt}^2 + \sigma_{\chi t}^2 - 2 \frac{1-\gamma}{\gamma} \alpha_{it} \sigma_{Rt} \sigma_{\chi t} \right) \right\}. \quad (10)$$

The first-order conditions of this problem gives the households' optimal consumption rate c_{Ht} and their optimal equity share α_{Ht} :

$$c_{Ht} = \rho^\psi \chi_t^{1-\psi}, \quad (11)$$

$$\alpha_{Ht} = \frac{1}{\gamma} \frac{\mu_{Rt} - r_t}{\sigma_{Rt}^2} + \frac{1-\gamma}{\gamma} \frac{\sigma_{\chi t}}{\sigma_{Rt}}. \quad (12)$$

Markov equilibrium. Households and entrepreneurs' policy functions are linear in wealth. This implies that the distribution of wealth within each group does not affect prices: only the distribution of wealth across groups matters for aggregate demand. Accordingly, I look for a Markovian equilibrium where the (endogenous) state variable is the share of aggregate wealth owned by entrepreneurs: $x_t = \int_{i \in \mathbb{I}_{Et}} W_{it} di / \left(\int_{i \in \mathbb{I}_t} W_{it} di \right)$.

Note that the market clearing equations (7) and (8) can be rewritten in terms of x_t :

$$x_t \rho_E + (1-x_t) c_{Ht} = \frac{1}{p_t}, \quad (7')$$

$$x_t \alpha_{Et} + (1-x_t) \alpha_{Ht} = 1. \quad (8')$$

The first equation says that the wealth-weighted average consumption rate equals the income yield of the tree while the second equation says that the wealth-weighted average equity share equals one.

We have five unknown functions of x : $r_t = r(x_t)$, $p_t = p(x_t)$, $\chi_{Ht} = \chi_H(x_t)$, $\alpha_{Ht} = \alpha_H(x_t)$, and $c_{Ht} = c_H(x_t)$. The market clearing equations (7') and (8') and the optimization conditions for households (11), (12), and (10) constitute a system of five equations. To solve for the equilibrium, it remains to obtain the law of motion of the endogenous state variable x_t using Ito's lemma:

Lemma 1. *The law of motion of x_t is given by*

$$\begin{aligned} dx_t &= \mu_{xt} dt + \sigma_{xt} dZ_t, \text{ where} \\ \mu_{xt} &\equiv x_t(1-x_t) \left((\alpha_{Et} - \alpha_{Ht})(\mu_{Rt} - r_t) + c_{Ht} - c_{Et} - (\alpha_{Et} - \alpha_{Ht})\sigma_{Rt}^2 + (\eta + \delta + \phi) \left(\frac{\pi}{x_t} - \frac{1-\pi}{1-x_t} \right) \right) \\ \sigma_{xt} &\equiv x_t(1-x_t)(\alpha_{Et} - \alpha_{Ht})\sigma_{Rt}. \end{aligned}$$

Moreover, the process x_t has a stationary distribution (i.e. neither $x_t = 0$ or $x_t = 1$ are absorbing states)

The drift of x_t is the sum of four terms: the difference in portfolio returns between entrepreneurs and households, the difference in their consumption rates, an Ito's term that accounts for the difference in their risk exposures, and a term related to the overlapping generation setting (i.e., due to population growth and death). This last term is ensures that the boundaries $x_t = 0$ and $x_t = 1$ are not absorbing states.

3.3 Characterizing the Wealth Distribution

I now characterize the distribution of wealth in this economy. The key theoretical contribution of this section is to obtain a simple formula for the tail index of the wealth distribution.

Normalized wealth. Since the economy grows over time, it is easier to work with individual wealth *normalized* by the average wealth in the economy: $w_{it} = W_{it}/(p_t Y_t)$. Applying Ito's lemma gives the following law of motion of normalized wealth for households i in group $j \in \{E, H\}$:

$$\begin{aligned} \frac{dw_{it}}{w_{it}} &= \mu_{wjt} dt + \sigma_{wjt} dZ_t + \nu_{wjt} dB_{it}, \text{ where} \\ \mu_{wjt} &\equiv r_t + \alpha_{jt}(\mu_{Rt} - r_t) - c_{jt} - g - \mu_{pt} - \sigma\sigma_{pt} - (\alpha_{jt} - 1)\sigma_{Rt}^2 \\ \sigma_{wjt} &\equiv (\alpha_{jt} - 1)\sigma_{Rt} \\ \nu_{wjt} &\equiv 1_{j=E}\alpha_{Et}V. \end{aligned} \tag{13}$$

Expression for the ζ -th moment. I study the dynamics of the density of the logarithm of normalized wealth in Appendix C.2. In the main text, however, I focus on characterizing its *Laplace*

transform, as it allows me to obtain the tail index of the wealth distribution in a more direct way. More precisely, I focus on characterizing the ξ -th cross sectional moment of wealth:

$$m_t(\xi) \equiv \frac{1}{|\mathbb{I}_t|} \int_{\mathbb{I}_t} w_i^\xi di. \quad (14)$$

As noted in [Gabaix et al. \(2016\)](#) and [Luttmer \(2016\)](#), the function $\xi \rightarrow m_t(\xi)$ can be seen as the Laplace transform of the density of logarithmic wealth at time t . Varying ξ allows one to examine the shape of different part of the distribution, with higher ξ reflecting the shape of the right tail of the wealth distribution.

The ξ th moment of normalized wealth is the population average of the ξ -th moment within each type of agents:

$$m_t(\xi) = \pi m_{Et}(\xi) + (1 - \pi) m_{Ht}(\xi),$$

where $m_{jt}(\xi) \equiv \left(\int_{\mathbb{I}_{jt}} w_i^\xi di \right) / |\mathbb{I}_{jt}|$ denotes the ξ th moment within each type $j \in \{E, H\}$. In turn, the following lemma gives an analytical expression for the law of motion of $m_{jt}(\xi)$:

Lemma 2. *The law of motion of the ξ th cross-sectional moment within group j is*

$$\begin{aligned} dm_{jt}(\xi) = & \left(\xi \mu_{wjt} + \frac{1}{2} \xi (\xi - 1) (\sigma_{wjt}^2 + \nu_{wjt}^2) \right) m_{jt}(\xi) dt + \xi \sigma_{wjt} m_{jt}(\xi) dZ_t \\ & + (\eta + \delta) \left(\left(\frac{\eta + \delta + \phi}{\eta + \delta} \right)^\xi - m_{jt}(\xi) \right) dt. \end{aligned} \quad (15)$$

This law of motion is composed of two main terms: the first line corresponds to the law of motion of the ξ -th moment among surviving agents (which is obtained using Ito's lemma), while the second line corresponds to the contribution of demographic changes: it is the difference between the ξ th moment within agents entering the economy minus the ξ -th moment within agents exiting the economy.

Since the cross-sectional ξ -th moment of normalized wealth obeys a linear stochastic differential equation (SDE), one can integrate it over time to obtain an analytical expression for $m_{jt}(\xi)$:

Proposition 1. *The ξ th cross-sectional moment within group j can be written in an integral form:*

$$m_{jt}(\xi) = \left(\int_{-\infty}^t (\eta + \delta) e^{\int_s^t \left(\xi (\mu_{wju} - \frac{1}{2} \sigma_{wju}^2) + \frac{1}{2} \xi (\xi - 1) \nu_{wju}^2 - (\eta + \delta) \right) du + \xi \sigma_{wju} dZ_u} ds \right) \left(\frac{\eta + \delta + \phi}{\eta + \delta} \right)^\xi.$$

This formula is intuitive: $m_{jt}(\xi)$ can be written as the population-weighted average of the ξ -th moment within each cohort. In turn, the cross-sectional moment within each cohort is obtained by noting that normalized wealth growth since birth follows a log-normal distribution. This result fully characterizes the cross-sectional ξ -th moment of wealth and therefore the wealth distribution.

Right tail. I now use these results to characterize the right tail of the wealth distribution. By analogy with the case of a static distribution, I define the tail index of a stochastic distribution as the smallest number ζ for which moments of order higher than ζ converge to infinity:

Definition 1 (Tail index). *The tail index of a sequence of distributions is the smallest positive number ζ such that moments of order higher than ζ converge to infinity almost surely as time goes to infinity.*

$$\zeta = \inf \left\{ \zeta \in \mathbb{R}_+ \cup \{+\infty\} \text{ s.t. } \lim_{t \rightarrow \infty} m_t(\zeta) = +\infty \text{ a.s.} \right\}$$

When $\zeta < +\infty$, we say that the distribution has a thick tail.

Armed with this definition, we can use the expression for $m_{jt}(\zeta)$ in Proposition 1 to obtain a simple characterization for the tail index of the wealth distribution: $m_{jt}(\zeta)$ converges to infinity a.s. whenever the term in the exponential averages to a positive number.

Proposition 2 (Tail index). *For $j \in \{H, E\}$, denote ζ_j the positive root of the quadratic equation:¹⁴*

$$\zeta_j \mathbb{E} \left[\mu_{wjt} - \frac{1}{2} \sigma_{wjt}^2 \right] + \frac{1}{2} \zeta_j (\zeta_j - 1) \mathbb{E} \left[v_{wjt}^2 \right] - (\eta + \delta) = 0, \quad (16)$$

where \mathbb{E} denotes the expectation with respect to the stationary density of x . Then,

1. The tail index of the wealth distribution within type $j \in \{E, H\}$ is ζ_j .
2. The tail index of the overall distribution is $\min(\zeta_E, \zeta_H)$.
3. $\min(\zeta_E, \zeta_H) > 1$; that is, the wealth distribution has a right tail thinner than Zipf's law.

In the rest of this section, I assume that $\zeta_H > \zeta_E$, which is the case as long as households do not grow too fast relative to entrepreneurs.¹⁵ In particular, this will be the case in the calibrated model in the next section. This implies that the relative proportion of entrepreneurs converges to one as wealth tends to infinity.¹⁶

¹⁴In case $\mathbb{E} \left[v_{wjt}^2 \right] = 0$, this is not a quadratic equation anymore. In this case, the solution should be understood as the limit of the positive root as $\mathbb{E} \left[v_{wjt}^2 \right] \rightarrow 0$; that is,

$$\zeta_j = \begin{cases} \frac{-\mathbb{E} \left[\mu_{wjt} - \frac{1}{2} \sigma_{wjt}^2 - \frac{1}{2} v_{wjt}^2 \right] + \sqrt{\mathbb{E} \left[\mu_{wjt} - \frac{1}{2} \sigma_{wjt}^2 - \frac{1}{2} v_{wjt}^2 \right]^2 + 2 \mathbb{E} \left[v_{wjt}^2 \right] (\eta + \delta)}}{2 \mathbb{E} \left[v_{wjt}^2 \right]} & \text{if } \mathbb{E} \left[v_{wjt}^2 \right] > 0, \\ \frac{\delta + \eta}{\mathbb{E} \left[\mu_{wjt} - \frac{1}{2} \sigma_{wjt}^2 \right]} & \text{if } \mathbb{E} \left[v_{wjt}^2 \right] = 0 \text{ and } \mathbb{E} \left[\mu_{wjt} - \frac{1}{2} \sigma_{wjt}^2 \right] > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

¹⁵Formally, $\zeta_H > \zeta_E$ iff $\mathbb{E} \left[\mu_{wHt} - \frac{1}{2} \sigma_{wHt}^2 \right] < \frac{\eta + \delta}{\zeta_E}$. This is true as long as $\mathbb{E} \left[\mu_{wHt} - \frac{1}{2} \sigma_{wHt}^2 \right] < \mathbb{E} \left[\mu_{wEt} - \frac{1}{2} \sigma_{wEt}^2 \right]$.

¹⁶See, also Gabaix (2009).

To understand the analytical characterization of the tail index given in (16), it is useful to discuss it in the context of the existing literature. As shown originally by [Reed \(2001\)](#), in a static economy in which individual wealth follows a geometric diffusion with drift μ , idiosyncratic volatility ν , and death rate δ , the wealth distribution has a tail index given by the positive root of¹⁷

$$\zeta\mu + \frac{1}{2}\zeta(\zeta - 1)\nu^2 - \delta = 0.$$

[Proposition 2](#) extends this fundamental result along two dimensions; that is, to an economy in which the dynamics of individual wealth *varies over time* and *is exposed to aggregate shocks*. It says that the same formula holds true after substituting ν^2 with the (time) average of the cross-sectional variance of wealth growth $\mathbb{E} \left[\nu_{wjt}^2 \right]$ and μ with the (time) average logarithmic growth of the cross-sectional average wealth of agents $\mathbb{E} \left[\mu_{wjt} - \frac{1}{2}\sigma_{wjt}^2 \right]$. In particular, the presence of aggregate shocks implies an adjustment of the drift of agents by an Ito's term, $-\frac{1}{2}\mathbb{E} \left[\sigma_{wjt}^2 \right]$. In the context of my model, this implies that even though entrepreneurs earn higher returns on average due to their higher equity share, the associated risk exposure has a dampened effect on the tail index as it decreases their average logarithmic growth rate.

It is also useful to connect this result to the theoretical literature about long-run survival in heterogeneous-agent models. As show in [Blume et al. \(1992\)](#), in infinite-horizon economies, a stationary wealth distribution obtains only when the average logarithmic growth rate of wealth for all agents is zero. While overlapping generation models break this equivalence, [Proposition 2](#) shows that this quantity is still key to determine the tail index of the wealth distribution in these models.

I conclude this section by giving an alternative interpretation of [Proposition 2](#). Dividing (16) by ζ_j gives

$$\mathbb{E} \left[\mu_{wjt} - \frac{1}{2}\sigma_{wjt}^2 \right] + \frac{1}{2}(\zeta_j - 1)\mathbb{E} \left[\nu_{wjt}^2 \right] - \frac{1}{\zeta_j}(\eta + \delta) = 0. \quad (17)$$

One can interpret this equation as a balance equation for the average normalized wealth in a top percentile (i.e., the top percentile wealth share); that is, the equation says that the average logarithmic growth of top wealth shares must be zero in equilibrium. Indeed, the first term in the left-hand side, $\mathbb{E} \left[\mu_{wjt} - \frac{1}{2}\sigma_{wjt}^2 \right]$, corresponds to the (time average) logarithmic growth of the average wealth of agents initially in the top percentile (“within” term), while the two other terms account for the growth in top wealth shares due to composition changes in the top percentile (“between” and “demography” terms discussed in [Gomez \(2018\)](#)).

In short, [Proposition 2](#) gives a tight relationship between the wealth distribution's tail index and the logarithmic growth of the average wealth of agents in a top percentile. This will be useful when calibrating the model because it will help me to discipline the consumption rate of en-

¹⁷See [Gabaix \(2008\)](#) and [Piketty and Zucman \(2015\)](#) for literature reviews.

trepreneurs from the tail index of the wealth distribution.

3.4 Extensions

To focus on the intuition, I have described a very parsimonious model with heterogeneous-agents. I now describe a few extensions of the model which would make it more realistic. The key message of this section is that these extensions would not affect (i) equilibrium asset prices (ii) the tail index of the wealth distribution and the elasticity of top wealth shares to stock market returns. This is the main reason why I will focus on these moments when calibrating the model in the next section.

Heterogeneity in initial endowment. In the model, all newborns are born with the same level of initial wealth. However, the model could be easily extended to allow newborns to be endowed with trees of different sizes (reflecting heterogeneous skills or heterogeneous inheritances).

In Appendix C.3.1, I show that this additional dimension of heterogeneity would not affect the tail index of the wealth distribution — at least as long as the distribution of initial wealth has a right tail that is “thin” enough, which is the case empirically. Moreover, it would not affect asset prices since the wealth distribution within each group of agents is irrelevant for asset prices (as agents have homothetic preferences).

Labor and capital income. In the model, an individual only earns one type of income. In reality, individuals earn both labor and dividend income. The distinction does not matter for household decisions as long as individuals have free access to financial markets (see, for instance, [Gârleanu and Panageas \(2015\)](#)).

Still, the distinction matters when mapping the model to the data. Wealth, in the model, corresponds to the capitalized value of all future income (i.e. “total wealth”), while, in the data, it only corresponds to the capitalized value of all dividend income (i.e. “financial wealth”).¹⁸

In Appendix C.3.2, however, I argue that this distinction does not really matter for the tail index of the wealth distribution or the elasticity of top wealth shares to stock market returns. This mainly comes from the fact that the difference between “financial wealth” and “total wealth” is small at the top of the wealth distribution, as most of the income received by agents in the top corresponds is capitalized in their wealth.

Hand-to-mouth households. In the model, households can freely trade in financial markets. In reality, a lot of households face financial frictions. To account for this fact, the model could be extended to assume that a fraction of households simply consumes the income they are endowed with every period. These households would not matter for asset prices as they do not trade assets. Similarly, they would not affect the elasticity of top wealth shares to stock returns or the tail index

¹⁸See, for instance, [Catherine et al. \(2020\)](#) and [Greenwald et al. \(2021\)](#).

of the wealth distribution as they do not appear in top percentiles. In this case, the model only describes the subset of households that freely trade in financial markets.

4 Quantitative Analysis

I now turn to the quantitative implications of the model. Section 4.1 presents the calibration, Section 4.2 discusses the feedback effect between wealth inequality and asset prices at the heart of the model, Section 4.3 evaluates the fit of the model with respect to moments untargeted by the calibration, and Section 4.4 studies analytically the impulse response of individual wealth to aggregate shocks.

4.1 Calibration

The model has 12 parameters, which I calibrate to match moments related to the U.S. economy.

Endowment Process. I start by calibrating five parameters related to endowment process $(g, \sigma, \delta, \eta, \phi)$. The drift and volatility of the endowment process per capita are chosen to match the growth and standard deviation of time-averaged annual consumption, as in Gârleanu and Panageas (2015); that is, $g = 2\%$ and $\sigma = 4\%$. The population growth rate η is chosen to match the annual growth of the number households in the U.S. since 1916; that is, $\eta = 1.5\%$. The death rate δ is chosen to match the annual death rate of households in the top 0.5% measured by Kopczuk and Saez (2004); that is, $\delta = 2.5\%$.¹⁹ The depreciation rate of trees is chosen to match the difference of 2% between the growth rate of dividends in the economy ($g + \eta$ in the model) and the dividend growth of existing firms ($g - \phi$ in the model) measured by Gârleanu et al. (2015), which gives $\phi = 0.5\%$.

Wealth Dynamics of Entrepreneurs. I use the theoretical and empirical results obtained in the previous sections to calibrate three parameters related to the wealth dynamics of entrepreneurs $(\alpha_E, \nu, \rho_E, \pi)$. I start by calibrating the share of wealth invested in equity by entrepreneurs, α_E , to match the regressions of top wealth shares on stock market returns from Section 2. To interpret these regressions, remember that in the model, agents only trade all-equity firms. In reality, firms issue a mix of debt and equity, and therefore stock market equity corresponds to a levered claim on the underlying firms. More precisely, following Modigliani-Miller logic, the model predicts an expected return of levered equity of $r_t + \lambda(\mu_{Rt} - r_t)$ and a volatility of $\lambda\sigma_{Rt}$, where λ denotes the average firm leverage (market value of assets over market value of equity). As a result, regressing aggregate wealth on levered equity in the model estimates $1/\lambda$, while regressing the wealth of households at the top of the wealth distribution on levered equity estimates α_E/λ . Therefore, the results in Table 1 imply $\lambda = 2.2$ and $\alpha_E = 2$. This estimate is high but it is a bit lower than typical

¹⁹This is roughly consistent with the 2.2% annual death rate in Forbes 400 measured in Gomez (2018).

calibrations in asset pricing models: in particular, [Gârleanu and Panageas \(2015\)](#) calibration implies that households at the top of the wealth distribution have an average exposure to aggregate shocks $\alpha_E \approx 2.5$, while [Di Tella \(2016\)](#) calibration implies that the average exposure of financial intermediaries to aggregate shocks is $\alpha_E \approx 2.8$.

The idiosyncratic volatility of trees is chosen to match the 20% annual cross-sectional dispersion of the wealth growth for agents at the top of the wealth distribution ($\sqrt{\mathbb{E}[\alpha_{Et}^2 \nu^2]}$ in the model), as measured in [Angeletos \(2007\)](#), [Benhabib et al. \(2011\)](#), and [Gomez \(2018\)](#); that is, $\nu = 10\%$.

I then calibrate the entrepreneur consumption rate, ρ_E , to match the tail index of the wealth distribution. I do so in two steps. First, I use Proposition 2 to infer the average logarithmic growth of entrepreneurs relative to the economy from the tail index of the wealth distribution. As discussed in Section 3, this step can be seen as deducing the logarithmic average wealth growth of agents in a top percentile from the fact that the logarithmic growth of top wealth shares is zero on average. [Klass et al. \(2006\)](#) and [Vermeulen \(2018\)](#) measure a power law exponent for the wealth distribution of $\zeta = 1.5$. Graphically, Appendix Figure A2 confirms their result by plotting the log percentile as a function of the log net worth for the U.S. distribution, in the SCF and the Forbes 400 data. Plugging this number into (17) and combining it with the calibration for δ, η, ν gives an estimate for the average logarithmic growth of entrepreneurs relative to the economy of 1.7%.²⁰

Second, I use this estimate to infer the consumption rate of entrepreneurs. Indeed, the average logarithmic growth of entrepreneurs' normalized wealth can be written as the difference between the average logarithmic growth of entrepreneurs and the average logarithmic growth of the economy:²¹

$$\mathbb{E} \left[\mu_{wEt} - \frac{1}{2} \sigma_{wEt}^2 \right] = \underbrace{\mathbb{E} \left[r_t + \alpha_{Et} (\mu_{Rt} - r_t) - \frac{1}{2} \alpha_{Et}^2 \sigma_{Rt}^2 \right]}_{\text{Average logarithmic return of entrepreneurs}} - \rho_E - \underbrace{\left(g - \frac{1}{2} \sigma^2 \right)}_{\text{Average logarithmic growth of economy}}.$$

The average logarithmic growth of the economy of 1.9%. To obtain an estimate for the average logarithm return of entrepreneurs, I use the average risk-free rate $\mathbb{E}[r_t] = 2.8\%$, average equity premium $\lambda \mathbb{E}[\mu_{Rt} - r_t] = 5.2\%$, and average variance $\lambda^2 \mathbb{E}[\sigma_{Rt}^2] = 0.03$ of the stock market from [Shiller \(2015\)](#), which gives an average logarithm return of 6.2%. Overall, this implies a consumption rate of entrepreneurs $\rho_E = 2.5\%$.

I now turn to the population share of entrepreneurs. I calibrate a value $\pi = 5\%$ to match the proportion of households that report that more than two-thirds of their wealth is invested in an actively managed business (S-corp or partnership) in the SCF in 2017. Note that the choice of

²⁰Note that in the model, this implies that the ratio between the average wealth of an entrepreneur and the average wealth of a newborn entrepreneurs is $(\delta + \eta) / (\delta + \eta - (\mathbb{E}[\mu_{wEt} - \frac{1}{2} \sigma_{wEt}^2])) \approx 1.7$, which is realistic.

²¹This comes from (13), after noting that the average drift of log asset prices is zero due to stationarity; that is, $\mathbb{E}[\mu_{pt} - \frac{1}{2} \sigma_{pt}^2] = 0$.

this parameter is a bit difficult since, in reality, there exists a continuum between households and entrepreneurs. Fortunately, Appendix Table A4 shows that the implication of the model for asset prices is not particularly sensitive to the choice of π .

Household Preferences. The model has three remaining parameters which correspond to households' preference parameters: their SDR ρ , their EIS ψ , and their RRA γ . I calibrate these three parameters jointly to match four asset price moments: the average and standard deviation of the risk-free rate and of stock market returns as reported in Shiller (2015).

Formally, denote $\theta \equiv (\rho, \pi, \gamma)$ the vector of parameters and $m(\theta)$ the vector of moments implied by these parameters; that is,

$$m(\theta) \equiv \left(\mathbb{E} [r_t], \sqrt{\mathbb{E} [r_t^2] - \mathbb{E} [r_t]^2}, \lambda \mathbb{E} [\mu_{Rt} - r_t], \lambda \mathbb{E} [\sigma_{Rt}] \right).$$

I pick the vector of parameters $\hat{\theta}$ which minimizes the distance $(\hat{m} - m(\theta))' \hat{W} (\hat{m} - m(\theta))$, where \hat{m} denotes the four moments in the data and \hat{W} is a diagonal matrix with the inverse variance of each moment measured in the data. For the sake of realism, I only search for an RRA γ and an inverse EIS $1/\psi$ below 20, as well as a SDR ρ below 10%.²² I search for the minimizer $\hat{\theta}$ globally using a differential evolution search algorithm.

Table 2 reports the estimated parameters. The table shows a high SDR ($\rho = 10\%$), a high RRA ($\gamma = 14$), and a low EIS ($\psi = 0.05$). The low EIS is consistent with evidence from the micro data for the average household (Vissing-Jørgensen, 2002; Best et al., 2020).

Table 3 reports the implied moments about asset price. The model can approximately match the four asset pricing moments used in the estimation: the model generates an interest rate of 3.7% (versus 2.8% in the data) with a low volatility of 0.8% (versus 0.9% in the data). Importantly, the model generates a high equity premium of 5% (versus 5.2% in the data) with a high standard deviation of equity returns of 16.3% (versus 18.2% in the data).

I report the solution of the model in Figure 3. Figure 3a plots the drift and volatility of the state variable while Figure 3b plots its associated stationary density. Figure 3c plots the risk-free rate $r(x)$ and expected return on equity $\mu_R(x)$. Finally, Figure 3d plots the price-to-income ratio $p(x)$ and the wealth-to-consumption ratio $c_H(x)$ of households. One can see that a higher share of wealth owned by entrepreneurs, x , is associated with a higher price-to-income ratios $p(x)$, as well as lower expected returns. The intuition is that, as more of the economy is owned by entrepreneurs, the aggregate demand for assets increases, which increases equilibrium valuations and decreases expected returns.

²²Remember that the SDR is the sum of a true impatience rate and a 2.5% hazard rate of death.

Table 2: Parameters

Description	Symbol	Value	Target
<i>Endowment Process</i>			
Endowment growth rate	g	2%	Per capita growth rate of consumption
Endowment volatility	σ	4%	SD of time-averaged consumption
Population growth rate	η	1.5%	Growth rate number U.S. households
Death hazard rate	δ	2.5%	Death rate in top 0.5% (Kopczuk and Saez, 2004)
Tree depreciation rate	ϕ	0.5%	Growth rate existing firms (Gârleanu et al., 2015)
<i>Entrepreneurs' dynamics</i>			
Entrepreneur equity share	α_E	2	Regression growth top 0.01% wealth on stock returns
Tree idiosyncratic volatility	ν	10%	Dispersion wealth growth at the top (Angeletos, 2007)
Entrepreneur SDR	ρ_E	2.5%	Tail index of wealth distribution
Entrepreneur population share	π	5%	Percentage households with $> 2/3$ of wealth in active business
<i>Household preferences</i>			
Household SDR	ρ	10%	Asset Price Moments
Household EIS	ψ	0.05	Asset Price Moments
Household RRA	γ	14	Asset Price Moments

Notes: This table summarizes the calibration discussed in Section 4.1. Each parameter is given at the annual frequency.

Table 3: Moments

Moments	Data	Model
<i>Targeted moments</i>		
Equity premium	5.2%	5.0%
Standard deviation market return	18.2%	16.3%
Average interest rate	2.8%	3.7%
Standard deviation interest rate	0.9%	0.8%
<i>Untargeted moments</i>		
Standard deviation log top 0.01% wealth share	0.40	0.28
Regression future excess returns on log top 0.01% wealth share	-0.11	-0.09

Notes: The second column reports the moments in the data while the third column reports the moments in the model. Each moment is given at the annual frequency.

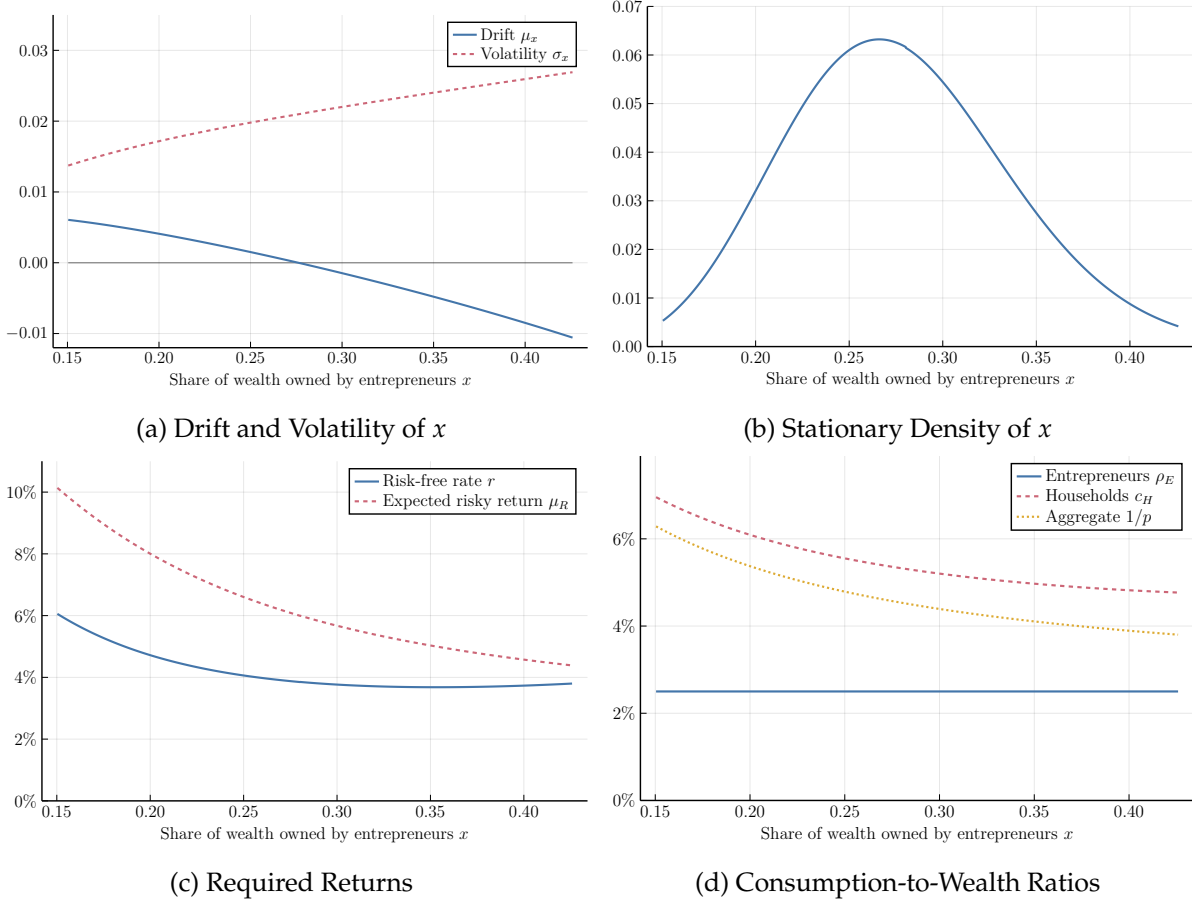


Figure 3: Estimated Model

Notes: This figure plots equilibrium quantities in the model in terms of the share of aggregate wealth owned by entrepreneurs, x_t . The bounds of the x-axis correspond to the 1% and the 99% quantiles of the state variable.

4.2 Inspecting the Mechanism

Volatility of equity returns. A key difficulty for asset pricing models is to generate highly volatile stock market returns despite the low volatility of the aggregate endowment (Campbell and Shiller (1988)). More precisely, in the data, the volatility of aggregate endowment is $\sigma \approx 4\%$ while the volatility of (unlevered) equity returns is $\sigma_R \approx 18\%/2.2 \approx 8\%$, which is twice as high.

The calibrated can generate such an “excess” volatility of equity returns due to a feedback loop between asset prices and wealth inequality. To see why, note that, on the one hand, the volatility of asset returns increases with the volatility of wealth inequality; indeed, (3) gives:

$$\begin{aligned} \sigma_R &= \sigma + \sigma_p \\ &= \sigma + \partial_x \log p \times \sigma_x, \end{aligned} \tag{18}$$

The volatility of returns, σ_R is the sum of the volatility of the aggregate endowment, σ and the volatility of asset valuations, σ_p . In turn, Ito’s lemma says that σ_p can be written as the product

of the semi-elasticity of asset valuations $\partial_x \log p$ and the volatility of the state variable, σ_x . In particular, note that a model with only one type of agent cannot match the excess volatility of asset prices, since, in this case, we have $\sigma_p = 0$, and, therefore, $\sigma_R = \sigma$.

On the other hand, Lemma 1 says that the volatility of wealth inequality increases with the volatility of asset returns:

$$\sigma_x = x(\alpha_E - 1)\sigma_R, \quad (19)$$

which reflects the fact that entrepreneurs tend to own more equity than the rest of the population.

These two equations summarize the feedback loop between wealth inequality and asset prices which is at the heart of the model. Following an aggregate income shock, the share of wealth owned by entrepreneurs increases (19), which, in turn, increases valuations (18), which then increases the share of wealth owned by entrepreneurs even more (19), etc. The outcome of this feedback loop can be found by combining (18) and (19) to solve for σ_R .²³

$$\sigma_R = \frac{1}{1 - (\alpha_E - 1)x\partial_x \log p} \times \sigma. \quad (20)$$

Overall, the volatility of stock market returns, σ_R , is equal to the volatility of aggregate income, σ , times a multiplier $1/(1 - (\alpha_E - 1)x\partial_x \log p)$. This multiplier increases with the heterogeneity in risk exposure across agents, $\alpha_E - 1$, as well as the elasticity of asset valuations to inequality, $x\partial_x \log p$.

Role of households preference parameters. I now discuss the role of household preferences in determining the semi-elasticity of valuations to wealth inequality, $\partial_x \log p$. Differentiating the market clearing condition for the goods market (7') gives that this semi-elasticity depends on the difference between the consumption rate of entrepreneurs and the consumption rate of households. Intuitively, the higher the heterogeneity in consumption rates between agents, the bigger the effect of a shift in the wealth distribution on the aggregate demand for assets, and, therefore, the bigger asset valuations have to adjust to clear the markets.

While the consumption rate of entrepreneurs is pinned down by the tail index of the wealth distribution, the consumption rate of households increases with ρ , $1/\psi$, and γ (due to precautionary savings). Increasing these parameters increases the heterogeneity in saving demands across agents, which makes asset valuations more sensitive to changes in the wealth distribution.

Consistent with this intuition, Appendix Table A4 reports the elasticity of asset price moments to household preference parameters. The table shows that increasing ρ , γ or $1/\psi$ increases the

²³Heuristically, this equation can be seen as the geometric sum of each round in this loop:

$$\sigma_R = \sigma + x\partial_x \log p(\alpha_E - 1)\sigma + (x\partial_x \log p \times (\alpha_E - 1))^2 \sigma + \dots$$

volatility of stock market returns while increasing the interest rate in the economy; this is because each parameter increases the consumption rate of households.

Appendix Table A4 also reports the sensitivity of asset price moments to α_E , the relative exposure of entrepreneurs to aggregate risk, ρ_E , their consumption rate, and π , their population share. The table shows that the high value of α_E and the low value of ρ_E measured in the data are essential to generate volatile stock market returns. As seen in (20), a high α_E is important in generating enough fluctuations in the wealth distributions while a low ρ_E is important for changes in the wealth distribution to affect equilibrium asset prices. In contrast, the value of π does not matter much for equilibrium asset prices.

A non-linear Campbell-Shiller decomposition. The previous discussion stresses that the semi-elasticity of valuations to wealth inequality, $\partial_x \log p$, is key to generate volatile asset returns.

One question is whether, in the calibrated model, this high elasticity reflects the effect of wealth inequality x on the risk-free rate or on excess equity returns. In the spirit of Campbell and Shiller (1988), I now derive an *exact* decomposition for the semi-elasticity of asset valuation to the state variable x

Proposition 3 (Decomposing valuation changes into changes in risk-free rates and expected future returns).

$$\begin{aligned} \partial_x \log p = & -\mathbb{E} \left[\underbrace{\int_0^\infty e^{-\int_0^t \frac{1}{p_s} ds} \frac{\partial x_t}{\partial x_0} \partial_x r(x_t) dt}_{\text{Risk-free rate channel}} \middle| x_0 = x \right] \\ & - \mathbb{E} \left[\underbrace{\int_0^\infty e^{-\int_0^t \frac{1}{p_s} ds} \frac{\partial x_t}{\partial x_0} \partial_x \left(\mu_R - \frac{1}{2} \sigma_R^2 - r \right) (x_t) dt}_{\text{Expected excess return channel}} \middle| x_0 = x \right], \end{aligned}$$

where $\frac{\partial x_t}{\partial x_0}$ denotes the stochastic derivative of the process (x_t) ; that is, the process which solves the stochastic differential equation:

$$\left(d \frac{\partial x_t}{\partial x_0} \right) / \left(\frac{\partial x_t}{\partial x_0} \right) = \partial_x \mu_x(x_t) dt + \partial_x \sigma_x(x_t) dZ_t, \quad (21)$$

with initial value 1 at $t = 0$.

This equation says that the effect of an increase in x on asset valuations, $\partial_x \log p$, is the sum of the effect of an increase in x on the path of future risk-free rate and of the effect on x on the path of future excess logarithmic equity returns $\mu_R - \frac{1}{2} \sigma_R^2 - r$. In contrast with Campbell and Shiller (1988), this decomposition is exact. Moreover, because of the continuous-time setup, each term can be computed analytically using the Feynman-Kac formula.²⁴

²⁴See the proof of 3 for more details.

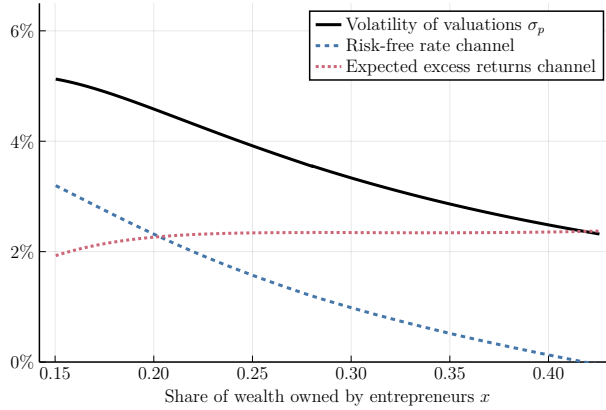


Figure 4: Decomposing the Volatility of Asset Valuations σ_p

Notes: This figure plots the volatility of asset valuations, $\sigma_p = \partial_x \log p \times \sigma_x$, as well as its decomposition into a “risk-free rate channel” and a “expected excess return channel” defined in Proposition 3. The bounds of the x-axis correspond to the 1% and the 99% quantiles of the state variable.

Figure 4 plots the obtained decomposition as a function of the state variable x : I find that the volatility of valuations mostly reflects changes in expected excess logarithmic returns rather than changes in future interest rates, which fits with the observation in Campbell and Shiller (1988). In the model, this happens because, as wealth is re-balanced to entrepreneurs, the aggregate demand for risk exposure increases, which means that fluctuations in asset prices also affect risk premia in equilibrium.

4.3 Untargeted Moments

I now discuss how the model matches two dimensions of the data which were not directly targeted by the calibration.²⁵

Standard deviation of Top 0.01% wealth share. I first ask whether the model can match the fluctuations of top wealth shares over time. In the data, the standard deviation of the logarithm of the top 0.01% is 0.4; in the model, the equivalent moment is 0.28 (Table 3), which is a bit lower. This difference is not surprising: as we already know from Section 2, stock market returns alone cannot fully account for the dynamics of top wealth shares over the 20th century (Figure 2). Still, it is reassuring that the model can generate sizable fluctuations in top wealth shares over time, due to the fact that it generates highly volatile stock returns with persistent top wealth shares.

Top 0.01% wealth share and future excess returns. I then examine a key implication in the model, which is that periods of high wealth inequality are associated with lower returns going forward (as seen in Figure 3c and Figure 4). The intuition is that, when more wealth is in the

²⁵Since these moments do not admit analytical expressions in the model, I obtain them by simulating the model for 500 years, using only the last 100 years to compute the moment in the simulated data. I then average the estimate across 500 simulations.

hands of entrepreneurs, the aggregate demand for equity is higher, which leads to lower expected returns in equilibrium.

One way to test this prediction is to regress future excess stock returns on the wealth share of the top 0.01%.²⁶

$$\log R_{M,t+1} - \log R_{f,t+1} = \alpha + \beta \log (\text{Wealth Share Top 0.01\%})_t + \epsilon_t, \quad (22)$$

where $\log R_{M,t}$ denotes the log stock market return, $\log R_{f,t}$ denotes the log risk-free rate.

Running this regression in the model gives an estimate $\beta \approx -0.09$. More interestingly, running this regression on U.S. data gives an estimate for $\beta \approx -0.11$ which is close and significant at the 10% level. It is well known that for a predictor that is persistent and correlated with returns, like top wealth shares, conventional t-statistics are misleading.²⁷ To address this concern, I rely on a test developed in [Campbell and Yogo \(2006\)](#), which is valid even when the predictor variable has a root close to or larger than one.²⁸ Appendix Table A5 reports that, even after potentially allowing for explosive dynamics in top wealth shares, the wealth share of the top 0.01% is found to significantly predict returns. Another way to correct for the persistence of the predictor is to use a detrended version of the predictor (see [Hodrick \(1992\)](#)): Appendix Table A5 also reports the regression using the five-year difference of the top wealth share in the right-hand side, which tends to strengthen its predictive power. Finally, Appendix Table A5 shows that the information in the wealth share of the top 0.01% is not subsumed by the price-dividend ratio, a widely known predictor of excess returns. These results suggest that fluctuations in wealth inequality generate important fluctuations in expected returns, which is consistent with the model.

In unreported results, I also find that fluctuations in top wealth inequality do not significantly predict fluctuations in the risk-free rate; this reflects the fact that the standard deviation of the interest rate is low in the model as in the data (Table 3).

4.4 Impulse Response of Individual Wealth

In the last section of the paper, I characterize the impulse response of individual wealth to aggregate income shocks. My main finding is that, while equilibrium changes in asset valuations amplify the short-run effect of aggregate shocks on entrepreneurs' wealth, they dampen their long-run effect.

To fix ideas, it is helpful to consider what would happen if asset valuations did not react to aggregate income shocks (i.e. $\sigma_p = 0$). Because aggregate income follows a random walk, and because entrepreneurs consume a constant fraction of their wealth, an aggregate income shock σdZ_t would permanently increase the normalized wealth of an entrepreneur $i \in \mathbb{I}_{Et}$ by $(\alpha_E -$

²⁶See [Toda and Walsh \(2016\)](#) for a related result using income, rather than wealth inequality.

²⁷For instance, [Elliott and Stock \(1994\)](#) and [Stambaugh \(1999\)](#).

²⁸The test can only be done for the restricted sample without gaps in the predictor, i.e. 1917-1951.

1) σdZ_t .²⁹

In the model, asset valuations increase after an aggregate income shock (i.e. $\sigma_p > 0$). In the short-run, this means that the normalized wealth of entrepreneurs “over-react” to aggregate income shocks: $(\alpha_E - 1)\sigma_R dZ_t > (\alpha_E - 1)\sigma dZ_t$. However, the effect of asset valuations is more subtle in the long-run. First, this response of valuations is purely temporary: a higher realized return today is associated with lower returns tomorrow. Second, higher valuations push entrepreneurs to consume relatively more than other agents in the economy, which tends to decrease their wealth in the long-run.

To formalize this point, I study the effect of an aggregate shock on the average wealth of an entrepreneur over the horizon $h > 0$. More precisely, I denote $\epsilon(x, h)$ the effect of an aggregate shock on the expected logarithm of the cross-sectional average normalized wealth of entrepreneurs who remain alive in h periods:^{30,31}

$$\epsilon(x_t, h) \equiv \sigma \left[\mathbb{E}_t \log \frac{1}{|\mathbb{I}_{E_t} \cap \mathbb{I}_{E_{t+h}}|} \int_{\mathbb{I}_{E_t} \cap \mathbb{I}_{E_{t+h}}} w_{E_{it+h}} di \right].$$

The next proposition gives an analytical expression for $\epsilon(x, h)$.

Proposition 4 (Analytical Impulse Response Function). *The effect of an aggregate shock on the average wealth of surviving entrepreneurs at horizon h is:*

$$\epsilon(x, h) = \sigma_{wE}(x) + \mathbb{E} \left[\int_0^h \frac{\partial x_t}{\partial x_0} \partial_x \left(\mu_{wE} - \frac{1}{2} \sigma_{wE}^2 \right) (x_t) dt \middle| x_0 = x \right] \sigma_x.$$

This equation decomposes the effect of an aggregate shock on wealth at horizon h into two terms.³² The first is the “instantaneous” effect of an aggregate shock on the normalized wealth of entrepreneurs, $\epsilon(x, 0) = \sigma_{wE}(x)$. The second term is the “indirect” effect of an aggregate shock through changes in the logarithmic growth rate of entrepreneurs’ wealth going forward.

The proof of the proposition (Appendix A) shows that the second term can be expressed as the solution of a linear ODE, which allows me to compute $\epsilon(x, h)$ numerically. Figure 5 plots the average impulse response function of the normalized wealth of an entrepreneurs following an aggregate shock, that is $h \rightarrow \mathbb{E}[\epsilon(x, h)]$. For reference, I also plot the impulse response of wealth in the absence of any valuation change; that is, $\mathbb{E}[(\alpha_E - 1)\sigma]$. This figure summarizes the main message of this section: endogenous asset prices amplify the effect of aggregate shocks on entrepreneurs’ wealth in the short-run but dampen their effects in the long-run.³³

Note, however, that the convergence towards the long run-effect is very slow. Even after ten years, it would be hard for an external observer to assess the dampening effect of higher valua-

²⁹That is, until they die and their wealth gets redistributed to newborns

³⁰Here, and in the rest of the paper, $\sigma[a_t]$ denotes the instantaneous volatility associated with the stochastic process a_t ; that is, $da_t = \mathbb{E}_t[da_t] + \sigma_t[a_t] dZ_t$.

³¹I examine the logarithm of the cross-sectional average wealth as it corresponds to the cumulative “within” term, as defined in Gomez (2018); that is the cumulative wealth growth of agents initially in a top percentile who survive over time. Alternatively, one could also examine the effect of an aggregate shock on the logarithm of individual wealth and

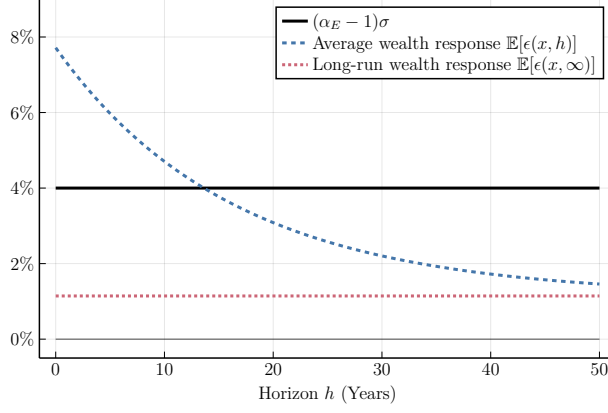


Figure 5: Impulse Response of Entrepreneur's wealth

Notes: The figure plots the average impulse response of (normalized) entrepreneur's wealth; that is, $h \rightarrow \mathbb{E}[\epsilon(x, h)]$. Since the model is calibrated at the annual frequency, this can be interpreted as the first-order response to a one-standard-deviation annual shock in the aggregate endowment.

tions. Appendix C.4 discusses more precisely the speed of convergence of $\epsilon(x, h)$ towards its limit, $\epsilon(x, \infty)$.

What determines the long-run effect of an aggregate shock on entrepreneurs' wealth? The following Corollary sheds light on this question.

Corollary 1 (Long-run Impulse Response). *The long-run effect of an aggregate shock on the average wealth of surviving entrepreneurs can be written as follows:*

$$\epsilon(x, \infty) = (\alpha_E - 1)\sigma + \mathbb{E} \left[\int_0^\infty \frac{\partial x_t}{\partial x_0} \partial_x \left(r + \alpha_E(\mu_R - r) - \frac{1}{2}\alpha_E^2\sigma_R^2 - c_E - \alpha_E(x_0) \left(\mu_R - \frac{1}{2}\sigma_R^2 - \frac{1}{p} \right) \right) (x_t) dt \middle| x_0 = x \right] \sigma_x.$$

This equation decomposes the long-run effect of an aggregate shock on entrepreneur's wealth into two terms. The first term, $(\alpha_E - 1)\sigma$, is a cash-flow effect: it corresponds to the long-run effect of an aggregate shock on entrepreneurs' wealth in the absence of any change in asset valuations. The second term, in contrast, isolates the long-run effect of the endogenous change in asset valuations.

This effect depends on the effect of a change in x on the average logarithmic growth rate of wealth of entrepreneurs minus α_E times the average logarithmic growth rate of wealth in the economy. In particular, this term is zero whenever $\alpha_E = 1$ and $c_E = 1/p$; intuitively, this reflects the fact that valuation changes have no long-run effect on wealth in representative agent economies.

In an heterogeneous-agent model, however, this term can be higher or lower than zero. In our particular setup, higher valuations push entrepreneurs to consume more, as a proportion of

obtain similar results.

³²See Borovička et al. (2014) and Alvarez and Lippi (2022) for related formula.

³³Appendix Figure A3 plots the impulse response of asset prices and expected returns .

their wealth, than the average agent in the economy; that is, $\partial_x(\alpha_E(x_0)/p - c_E) < 0$. This implies that higher valuations have a more negative effect on the growth rate of wealth for entrepreneurs than on the growth rate of wealth for the average agent in the economy. Overall, this means that the response of asset valuations tends to dampen the long-run effect of an aggregate shock on the wealth of entrepreneurs.³⁴

5 Conclusion

This paper examines theoretically and empirically the joint response of asset prices and wealth inequality to aggregate shocks. When an aggregate shock hits the economy, wealth inequality increases. As wealth is re-balanced towards agents with a higher demand for assets, asset valuations increase in equilibrium. This feedback loop between wealth inequality and asset prices magnifies effect of an aggregate shock on top wealth inequality in the short-run. However, in the longer-run, this mechanism helps to dissipate the effect of an aggregate shock on the wealth distribution, as it pushes wealthy households to consume relatively more.

Overall, my paper makes three distinct contributions. First, I document that households at the top of the wealth distribution are twice as exposed to equity returns as the rest of the distribution. Because top wealth shares tend to be persistent, this leads to large fluctuations in top wealth inequality over time. Second, I characterize the dynamics of the wealth distribution in a Markovian economy with aggregate shocks and I obtain a simple formula for its tail index, which depends on the average logarithmic return of households at the top of the wealth distribution. Third, I calibrate a heterogeneous-agent model on U.S. data to quantify the response of asset prices and wealth inequality to aggregate income shocks.

For simplicity, I only consider shocks to aggregate income in the model. However, the interplay I describe between asset prices and wealth inequality would also appear with shocks that redistribute aggregate income between labor and capital (Greenwald et al., 2016, Moll et al., 2021), shocks between young and old households (Gârleanu et al., 2012), or monetary policy shocks (Silva, 2016, Kekre and Lenel, 2021).

Finally, this interplay between wealth inequality and asset prices could have effects on real quantities as well, through changes in corporate investment policies or labor supply. Exploring these effects requires moving away from an endowment economy, which I leave for future research.

³⁴This comes from the fact that, in the model as in the data, agents at the top of the wealth distribution tend to have a higher elasticity of intertemporal substitution (EIS) than the rest of the population.

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A Proofs

A.1 Proofs

Proof of Lemma 1. Differentiating $x_t = \left(\int_{i \in \mathbb{I}_{Et}} W_{it} \, di \right) / \left(\int_{i \in \mathbb{I}_{Et}} W_{it} \, di + \int_{i \in \mathbb{I}_{Ht}} W_{it} \, di \right)$ with respect to time gives:³⁵

$$dx_t = x_t(1 - x_t) \left(\frac{d \left(\int_{i \in \mathbb{I}_{Et}} W_{it} \, di \right)}{\int_{i \in \mathbb{I}_{Et}} W_{it} \, di} - \frac{d \left(\int_{i \in \mathbb{I}_{Ht}} W_{it} \, di \right)}{\int_{i \in \mathbb{I}_{Ht}} W_{it} \, di} - \left(\frac{\sigma \left[\int_{i \in \mathbb{I}_{Et}} W_{it} \, di \right]}{\int_{i \in \mathbb{I}_{Et}} W_{it} \, di} - \frac{\sigma \left[\int_{i \in \mathbb{I}_{Ht}} W_{it} \, di \right]}{\int_{i \in \mathbb{I}_{Ht}} W_{it} \, di} \right) \frac{\sigma \left[\int_{i \in \mathbb{I}_{Et}} W_{it} \, di \right]}{\int_{i \in \mathbb{I}_{Et}} W_{it} \, di} dt \right). \quad (23)$$

In turn, the dynamics of aggregate wealth within each group $j \in \{E, H\}$ is given by:

$$\begin{aligned} d \left(\int_{i \in \mathbb{I}_{jt}} dW_{it} \, di \right) &= \int_{i \in \mathbb{I}_{jt}} dW_{it} \, di + (|\mathbb{I}_{jt}| / |\mathbb{I}_t|) (\eta + \delta + \phi) \left(\int_{i \in \mathbb{I}_t} W_{it} \, di \right) dt - \delta \left(\int_{i \in \mathbb{I}_{jt}} W_{it} \, di \right) dt \\ \Rightarrow \frac{d \left(\int_{i \in \mathbb{I}_{jt}} dW_{it} \, di \right)}{\int_{i \in \mathbb{I}_{jt}} W_{it} \, di} &= \frac{\int_{i \in \mathbb{I}_{jt}} dW_{it} \, di}{\int_{i \in \mathbb{I}_{jt}} W_{it} \, di} + \frac{\left(\int_{i \in \mathbb{I}_t} dW_{it} \, di \right) / |\mathbb{I}_t|}{\left(\int_{i \in \mathbb{I}_{jt}} W_{it} \, di \right) / |\mathbb{I}_{jt}|} (\eta + \delta + \phi) dt - \delta dt. \end{aligned}$$

Plugging this equation into the law of motion of x_t (23) gives

$$\begin{aligned} dx_t = x_t(1 - x_t) &\left(\frac{\int_{i \in \mathbb{I}_{Et}} dW_{it} \, di}{\int_{i \in \mathbb{I}_{Et}} W_{it} \, di} - \frac{\int_{i \in \mathbb{I}_{Ht}} dW_{it} \, di}{\int_{i \in \mathbb{I}_{Ht}} W_{it} \, di} + (\eta + \delta + \phi) \left(\frac{\pi}{x_t} - \frac{1 - \pi}{1 - x_t} \right) dt \right. \\ &\left. - \left(\frac{\int_{i \in \mathbb{I}_{Et}} \sigma [W_{it}] \, di}{\int_{i \in \mathbb{I}_{Et}} W_{it} \, di} - \frac{\int_{i \in \mathbb{I}_{Ht}} \sigma [W_{it}] \, di}{\int_{i \in \mathbb{I}_{Ht}} W_{it} \, di} \right) \sigma_{Rt} dt \right). \end{aligned}$$

Plugging the law of motion of W_{it} for households (4) and entrepreneurs (6) gives the result.

Finally, one can see that $\sigma_{xt}(0) = \sigma_{xt}(1) = 0$, with $\mu_{xt}(0) > 0$ and $\mu_{xt}(1) < 0$. As discussed in [Borovička \(Forthcoming\)](#), this ensures that the boundaries $x_t = 0$ and $x_t = 1$ are not absorbing states, and, therefore, that the process has a stationary wealth distribution. □

³⁵Here, and in the rest of the paper, $\sigma_t [a_t]$ denotes the instantaneous volatility associated with the stochastic process a_t ; that is, $da_t = E_t [da_t] + \sigma_t [a_t] dZ_t$.

Proof of Lemma 2. Differentiating $m_{jt}(\xi) = \left(\int_{i \in \mathbb{I}_{jt}} w_{it}^{\xi} \, di \right) / |\mathbb{I}_{jt}|$ for $j \in \{E, H\}$ with respect to time gives:

$$\begin{aligned}
dm_{jt}(\xi) &= m_{jt}(\xi) \left(\frac{d \left(\int_{i \in \mathbb{I}_{jt}} w_{it}^{\xi} \, di \right)}{\int_{i \in \mathbb{I}_{jt}} w_{it}^{\xi} \, di} - \frac{d |\mathbb{I}_{jt}|}{|\mathbb{I}_{jt}|} \right) \\
&= m_{jt}(\xi) \left(\frac{\int_{i \in \mathbb{I}_{jt}} dw_{it}^{\xi} \, di + (\eta + \delta) |\mathbb{I}_{jt}| \left(\frac{\eta + \delta + \phi}{\eta + \delta} \right)^{\xi} dt - \delta \left(\int_{i \in \mathbb{I}_{jt}} w_{it}^{\xi} \, di \right) dt}{\int_{i \in \mathbb{I}_{jt}} w_{it}^{\xi} \, di} - \eta dt \right) \\
&= m_{jt}(\xi) \frac{\int_{i \in \mathbb{I}_{jt}} dw_{it}^{\xi} \, di}{\int_{i \in \mathbb{I}_{jt}} w_{it}^{\xi} \, di} + (\eta + \delta) \left(\left(\frac{\eta + \delta + \phi}{\eta + \delta} \right)^{\xi} - m_{jt}(\xi) \right) dt. \tag{24}
\end{aligned}$$

In turn, the law of motion of w_{it}^{ξ} for $i \in \mathbb{I}_{jt}$ can be obtained by applying Ito's lemma on (13):

$$\frac{dw_{it}^{\xi}}{w_{it}^{\xi}} = \left(\xi \mu_{wjt} + \frac{1}{2} \xi (\xi - 1) (\sigma_{wjt}^2 + \nu_{wjt}^2) \right) dt + \xi \sigma_{wjt} \, dZ_t.$$

Plugging this equation into (24) gives the result □

Lemma 3 (Stability of Linear Functional). *Let $x_t \in \mathbb{R}$ be a continuous-time strong Markov process non-explosive, irreducible, positive recurrent with unique invariant probability measure. Consider the process*

$$dM_t = (\mu(x_t)M_t + b(x_t)) \, dt + \sigma(x_t)M_t \, dZ_t \tag{25}$$

with $\mathbb{P}(b(x) \geq 0) = 1$ and $\mathbb{P}(b(x) > 0) > 0$, and μ and σ are integrable.

(i) If $\mathbb{E}[\mu(x) - \frac{1}{2}\sigma(x)^2] > 0$, M_t converges to infinity a.s.

(ii) If $\mathbb{E}[\mu(x) - \frac{1}{2}\sigma(x)^2] < 0$, M_t does not converge to infinity a.s.

where \mathbb{E} denotes the expectation with respect to the invariant probability measure of x .

Proof of Lemma 3. The case $b(x_t) = 0$ (purely multiplicative process) is proven in [Maruyama and Tanaka \(1959\)](#). This lemma extends their result to the case $b(x_t) \neq 0$.

The proof proceeds in two steps. Following [Maruyama and Tanaka \(1959\)](#), I first bound the continuous-time process M_t by a discrete time process. I then characterize the asymptotic limit of the discrete time process using results from [Vervaat \(1979\)](#).

Step 1. For $\tau > 0$, we have the following recurrence eq:

$$M_{t+\tau} = e^{\int_t^{t+\tau} ((\mu(x_u) - \frac{1}{2}\sigma(x_u)^2) \, du + \sigma(x_u) \, dZ_u)} M_t + \int_t^{t+\tau} e^{\int_s^{t+\tau} ((\mu(x_u) - \frac{1}{2}\sigma(x_u)^2) \, du + \sigma(x_u) \, dZ_u)} b(x_s) \, ds.$$

Denote I the set of values that x_t can take. Take $a < b$, both in I . Define the sequence of stopping times $S_0 = 0$ and

$$\begin{aligned}
T_n &\equiv \inf\{t > S_n; x_t = a\}, \\
S_{n+1} &\equiv \inf\{t > T_n; x_t = b\}.
\end{aligned}$$

Define

$$\begin{aligned} X_n &= M_{T_n}, \\ A_n &= \exp \left(\int_{T_n}^{T_{n+1}} \left((\mu(x_u) - \frac{1}{2}\sigma(x_u)^2) du + \sigma(x_u) dZ_u \right) \right), \\ B_n &= \int_{T_n}^{T_{n+1}} e^{\int_s^{T_{n+1}} ((\mu(x_u) - \frac{1}{2}\sigma(x_u)^2) du + \sigma(x_u) dZ_u)} b(x_s) ds. \end{aligned}$$

Note that X_n bounds the continuous time process M_t : for $t \in (T_n, T_{n+1}]$, $X_n \leq M_t \leq X_{n+1}$. In particular, M_t converges to infinity a.s. if and only if X_n converges to infinity a.s.

Step 2. The sequence X_n satisfies the following recurrence relation:

$$X_{n+1} = A_n X_n + B_n,$$

where A_n and B_n are i.i.d over time. Moreover, A_1 is positive a.s., B_1 is non negative a.s. with $\mathbb{P}(B_1 > 0) > 0$ and $\mathbb{E}[\log B_1] < \infty$. As proven by [Vervaat \(1979\)](#), X_n converges in distribution if $\mathbb{E}[\log A_1] < 0$, i.e. $\mathbb{E} \left[\int_{T_1}^{T_2} (\mu(x_u) - \frac{1}{2}\sigma(x_u)^2) du \right] \geq 0$, and converges a.s. to infinity if $\mathbb{E}[\log A_1] > 0$. Finally, as shown in [Maruyama and Tanaka \(1959\)](#), any integrable function f , $\mathbb{E} \left[\int_{T_1}^{T_2} f(x_u) du \right] \geq 0$ iff $\mathbb{E}[f(x)] \geq 0$. This gives the result. \square

Proof of Proposition 2. I first prove the first claim. For $j \in \{E, H\}$, denote f_j the function defined as:

$$f_j : \zeta \rightarrow \zeta \mathbb{E} \left[\mu_{wjt} - \frac{1}{2}\sigma_{wjt}^2 \right] + \frac{1}{2}\zeta(\zeta - 1)\mathbb{E}[v_{wjt}^2] - (\eta + \delta). \quad (26)$$

In case $\mathbb{E}[v_{wjt}^2] > 0$, the function f_j is quadratic in ζ . Since $f_j(0) = -(\eta + \delta) < 0$ and $f_j(\zeta) \rightarrow +\infty$ as $\zeta \rightarrow +\infty$, there exists a unique $\zeta_j > 0$, such that $f_j(\zeta_j) = 0$. As $\mathbb{E}[v_{wjt}^2] \rightarrow 0$, this solution tends to a positive finite limit iff $\mathbb{E} \left[\mu_{wjt} - \frac{1}{2}\sigma_{wjt}^2 \right] > 0$; otherwise, it converges to $+\infty$.

In all cases, f_j is negative for $\zeta \in [0, \zeta_j)$ and positive for $\zeta \in (\zeta_j, +\infty)$. Applying Lemma 3 on $m_{jt}(\zeta)$, whose law of motion is given in (15), gives that $\zeta_j = \inf\{\zeta \in \mathbb{R}_+ \cup \{+\infty\} \mid \lim_{t \rightarrow \infty} m_{jt}(\zeta) = +\infty \text{ a.s.}\}$; that is, ζ_j corresponds to the tail index of the distribution within the group j .

I now turn to the second claim of the Proposition. Since $m_t(\zeta) = \pi m_{Et}(\zeta) + (1 - \pi)m_{Ht}(\zeta)$, and both quantities are always positive, $m_t(\zeta)$ converges to infinity a.s. iff either $m_{Et}(\zeta)$ or $m_{Ht}(\zeta)$ converges to infinity a.s. Therefore, the tail index of the wealth distribution is $\min(\zeta_E, \zeta_H)$.

The third claim is a direct consequence that, as proven in Lemma 1, x_t does not converge to 0 or 1 a.s. To understand better the intuition, note that Lemma 1 implies:

$$d \log x_t = \left(\mu_{wE} - \frac{1}{2}\sigma_{wE}^2 + (\eta + \delta) \left(\frac{\eta + \delta + \phi}{\eta + \delta} \frac{1}{x_t} - 1 \right) \right) dt + \sigma_{wE} dZ_t.$$

The fact that x_t has a stationary distribution implies $\mathbb{E}[d \log x_t] = 0$, which gives

$$\mathbb{E} \left[\mu_{wE} - \frac{1}{2}\sigma_{wE}^2 \right] = (\eta + \delta) \left(1 - \frac{\eta + \delta + \phi}{\eta + \delta} \mathbb{E} \left[\frac{1}{x_t} \right] \right).$$

This implies $f_j(1) = -(\eta + \delta + \phi) \mathbb{E} \left[\frac{1}{x_t} \right] < 0$, and, therefore, $\zeta_j > 1$. \square

Proof of Proposition 3. The definition of returns in (3) can be re-written in logs:

$$d \log R_{it} = \frac{1}{p_t} dt + d \log Y_{it} + d \log p_t.$$

Taking expectations at time t gives

$$\left(\mu_{Rt} - \frac{1}{2} \sigma_{Rt}^2 \right) dt = \left(\frac{1}{p_t} + \mu - \phi - \frac{1}{2} \sigma^2 \right) dt + \mathbb{E}_t [d \log p_t].$$

Differentiating with respect to x_t gives:

$$\frac{\partial x_t}{\partial x_0} \partial_x \left(\mu_{Rt} - \frac{1}{2} \sigma_{Rt}^2 \right) dt = -\frac{1}{p_t} \frac{\partial x_t}{\partial x_0} \partial_x \log p_t dt + \mathbb{E}_t \left[d \left(\frac{\partial x_t}{\partial x_0} \partial_x \log p_t \right) \right] \quad (27)$$

Integrating forward gives

$$\partial_x \log p(x) = -\mathbb{E} \left[\int_0^\infty e^{-\int_0^t \frac{1}{p_s} ds} \frac{\partial x_t}{\partial x_0} \partial_x \left(\mu_R - \frac{1}{2} \sigma_R^2 \right) (x_t) dt \middle| x_0 = x \right]$$

Splitting $\partial_x \left(\mu_R - \frac{1}{2} \sigma_R^2 \right)$ into $\partial_x (r)$ and $\partial_x \left(\mu_R - \frac{1}{2} \sigma_R^2 - r \right)$ gives the decomposition in the proposition.

For the sake of completeness, I also discuss how to compute each term in the decomposition numerically. Following [Hansen and Scheinkman \(2009\)](#), consider the semi-group of operators $(\mathcal{T}_t)_{t \geq 0}$ defined by

$$\mathcal{T}_t f = \mathbb{E} \left[\frac{\partial x_t}{\partial x_0} f(x_t) \middle| x_0 = x \right],$$

and the corresponding infinitesimal generator

$$\begin{aligned} \mathbb{T} f &= \lim_{t \rightarrow 0} \frac{1}{t} \mathbb{E} \left[\frac{\partial x_t}{\partial x_0} f(x_t) \middle| x_0 = x \right] \\ &= (\partial_x \mu_x) f + (\mu_x + \sigma_x \partial_x \sigma_x) \partial_x f + \frac{1}{2} \sigma_x^2 \partial_{xx} f \end{aligned} \quad (28)$$

Using the definition of \mathbb{T} in (28), we have, for any smooth function f :

$$\begin{aligned} \mathbb{E} \left[\int_0^\infty e^{-\int_0^t \frac{1}{p_s} ds} \frac{\partial x_t}{\partial x_0} f(x_t) dt \middle| x_0 = x \right] &= \left(\int_0^\infty e^{-t (\text{Diag} \left(\frac{1}{p} \right) - \mathbb{T})} dt \right) f(x) \\ &= \left(\text{Diag} \left(\frac{1}{p} \right) - \mathbb{T} \right)^{-1} f \end{aligned}$$

This means that, for any function f , we can obtain $g(x) = \mathbb{E} \left[\int_0^\infty e^{-\int_0^t \frac{1}{p_s} ds} \frac{\partial x_t}{\partial x_0} f(x_t) dt \middle| x_0 = x \right]$ as the solution of the linear ODE:

$$\left(\text{Diag} \left(\frac{1}{p} \right) - \mathbb{T} \right) g = f$$

This allows me to compute separately the “risk-free rate channel” and the “expected-return channel”. \square

Proof of Proposition 4. We have

$$\begin{aligned}
\epsilon(x_t, h) &= \sigma [\mathrm{d} \log w_{Et}] + \sigma [\mathrm{d} \mathbb{E}_t [\log \mathbb{E}_{t+h} [w_{Et+h}] - \log w_{Et}]] \\
&= \sigma_{wEt} + \partial_x \mathbb{E} \left[\int_t^{t+h} \left(\mu_{wE} - \frac{1}{2} \sigma_{wE}^2 \right) (x_s) \mathrm{d}s \middle| x_t = x_t \right] \sigma_{xt} \\
&= \sigma_{wE}(x) + \mathbb{E} \left[\int_0^h \frac{\partial x_t}{\partial x_0} \partial_x \left(\mu_{wE} - \frac{1}{2} \sigma_{wE}^2 \right) (x_t) \mathrm{d}t \middle| x_0 = x \right] \sigma_x.
\end{aligned}$$

I now derive an alternative expression for $\epsilon(x, h)$ which is useful to compute it numerically. Taking the limit $h \rightarrow \infty$ in the last equation gives:

$$\begin{aligned}
\epsilon(x, \infty) &= \sigma_{wE} + \mathbb{E} \left[\int_0^\infty \frac{\partial x_t}{\partial x_0} \partial_x \left(\mu_{wE} - \frac{1}{2} \sigma_{wE}^2 \right) (x_t) \mathrm{d}t \middle| x_0 = x \right] \sigma_x \\
&= \sigma_{wE} + \left(-\mathbb{T}^{-1} \partial_x \left(\mu_{wE} - \frac{1}{2} \sigma_{wE}^2 \right) \right) \sigma_x
\end{aligned}$$

where \mathbb{T} is infinitesimal generator defined in (28). Given this definition, we can rewrite $\epsilon(x, h)$ as:

$$\epsilon(x, h) - \epsilon(x, \infty) = \sigma_x e^{\mathbb{T}h} \sigma_x^{-1} (\epsilon(x, 0) - \epsilon(x, \infty)), \quad (29)$$

which says that the operator \mathbb{T} encodes the relaxation of $\epsilon(x, 0)$ towards $\epsilon(x, \infty)$.

In particular, this formula implies that we can solve numerically for $\epsilon(x, h)$ by solving the linear PDE

$$\partial_h \epsilon(x, h) = \sigma_x \mathbb{T} \sigma_x^{-1} (\epsilon(x, h) - \epsilon(x, \infty))$$

with initial boundary condition $\epsilon(x, 0) = (\alpha_E - 1) \sigma_R$. □

Proof of Corollary 1. The definition of μ_{wE} and σ_{wE} in (13) gives

$$\epsilon(x, \infty) = (\alpha_E - 1)(\sigma + \sigma_p) + \mathbb{E} \left[\int_0^\infty \frac{\partial x_t}{\partial x_0} \partial_x \left(r + \alpha_E(\mu_R - r) - c_E - \frac{1}{2} \alpha^2 \sigma_R^2 - \left(\mu_p - \frac{1}{2} \sigma_p^2 \right) \right) (x_t) \mathrm{d}t \middle| x_0 = x \right] \sigma_x.$$

Differentiating $\mathbb{E}_t [\mathrm{d} \log p_t] = \left(\mu_{pt} - \frac{1}{2} \sigma_{pt}^2 \right) \mathrm{d}t$ gives

$$\mathbb{E}_t \left[\mathrm{d} \left(\frac{\partial x_t}{\partial x_0} \partial_x \log p_t \right) \right] = \frac{\partial x_t}{\partial x_0} \partial_x \left(\mu_{pt} - \frac{1}{2} \sigma_{pt}^2 \right) \mathrm{d}t$$

Integrating forward gives

$$\partial_x \log p = -\mathbb{E} \left[\int_0^\infty \frac{\partial x_t}{\partial x_0} \partial_x \left(\mu_p - \frac{1}{2} \sigma_p^2 \right) (x_t) \mathrm{d}t \middle| x_0 = x \right].$$

Plugging this expression into the formula for $\epsilon(x, \infty)$ gives:

$$\begin{aligned}
\epsilon(x, \infty) &= (\alpha_E - 1)\sigma + \mathbb{E} \left[\int_0^\infty \frac{\partial x_t}{\partial x_0} \partial_x \left(r + \alpha_E(\mu_R - r) - \frac{1}{2} \alpha_E^2 \sigma_R^2 - c_E \right) (x_t) \mathrm{d}t \right. \\
&\quad \left. - \alpha_E(x_0) \int_0^\infty \frac{\partial x_t}{\partial x_0} \partial_x \left(\mu_p - \frac{1}{2} \sigma_p^2 \right) (x_t) \mathrm{d}t \middle| x_0 = x \right] \sigma_x.
\end{aligned}$$

Finally, note that the definition of returns (3) implies:

$$\partial_x \left(\mu_p - \frac{1}{2} \sigma_p^2 \right) = \partial_x \left(\mu_R - \frac{1}{2} \sigma_R^2 - \frac{1}{p} \right).$$

Plugging this result into the previous equation gives the result. □

Online Appendix

B Empirical Appendix

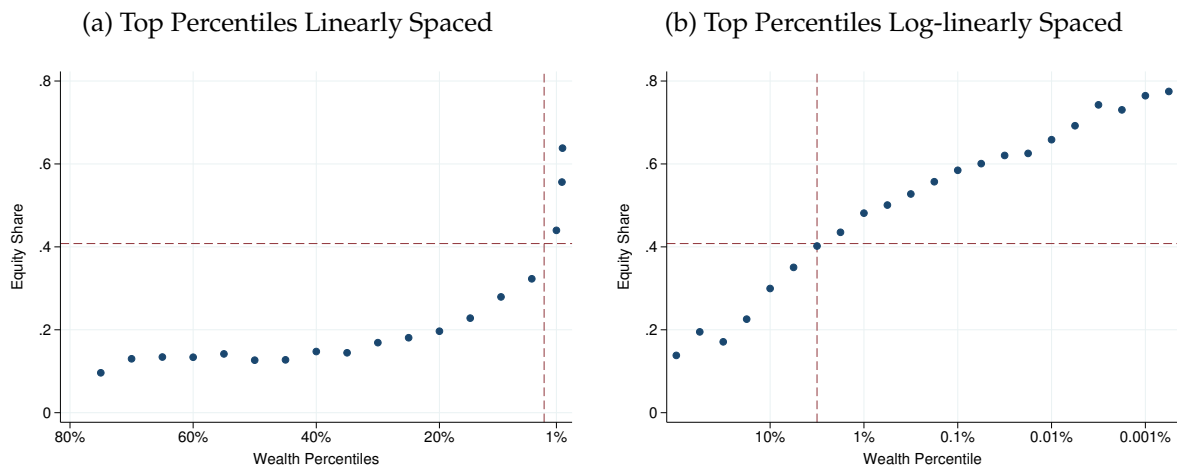
B.1 Portfolio Holdings in SCF

In the main text, I measure the exposure to equity return in a top percentile as the elasticity of the average wealth in top percentile to stock market returns. In this section, I show that measuring the equity exposure from portfolio holdings using the Survey of Consumer Finances (SCF) gives similar results.³⁶

Figure A1a plots the average equity share within percentile bins across the wealth distribution, where the equity share is defined as the total investment in equity over financial wealth. The average equity share masks a substantial heterogeneity across households. The equity share is essentially flat at 0.2 over the majority of the wealth distribution, but increases sharply within the top 1%. Figure A1b plots the equity share with respect to the *log* top percentiles, showing that the equity share is approximately linear in the log percentile at the top of the distribution: this suggests that the bulk of the heterogeneity is concentrated within the top percentiles.

Note that the average equity share is 0.4 while the equity share at the very top is 0.8. This is roughly consistent with the evidence in Table 1 showing that households at the top of the wealth distribution are twice as exposed as the average household to equity returns.

Figure A1: Portfolio Holdings in SCF



Notes. Figure A1a plots the average equity share within 20 linearly spaced percentile bins in the wealth distribution. Figure A1b plots the average equity share within 20 logarithmically spaced percentile bins in the wealth distribution. The horizontal line represents the average equity share. The vertical line splits the set of households in two: households on either side of the vertical line own half of total wealth (this corresponds to top percentile $\approx 3\%$).

Data from the Survey of Consumer Finance (SCF), a cross-sectional survey of US households from 1989 to 2013. The equity share is constructed as $(\text{equity} + \text{bus}) / \text{net worth}$.

A stylized fact in the household finance literature is that stock market participation increases with wealth (Vissing-Jørgensen (2002)). Therefore, the increase in the equity share within the top percentiles

³⁶The survey is a repeated cross-section of about 4,000 households per survey year, including a high-wealth sample. The survey is conducted every three years, from 1989 to 2013. The respondents provide information on their financial wealth, including their investments in public and private equity.

Table A1: Portfolio Holdings in SCF

	Groups of Households Defined by Wealth Percentiles			
	All Households	1% – 0.1%	0.1% – 0.01%	Top 0.01%
<i>Panel A: All Sample</i>				
Equity Share	40.8%	55.8%	65.8%	73.9%
Public Equity	20.2%	22.0%	21.1%	19.6%
Private Equity	20.6%	33.9%	44.6%	54.4%
Non Actively Managed	2.4%	4.5%	6.3%	7.8%
Actively Managed	18.2%	29.4%	38.4%	46.6%
<i>Panel B: Stockholders</i>				
Is Stockholder	45.9%	90.7%	91.2%	91.0%
Equity Share among Stockholders	44.7%	56.0%	65.9%	76.0%
<i>Panel C: Entrepreneurs</i>				
Is Entrepreneur	10.5%	62.1%	69.8%	78.5%
Equity Share among non-Entrepreneurs	26.8%	40.7%	50.0%	57.9%
<i>Panel D: Stock Options Holders</i>				
Received Stock Options	6.4%	11.2%	11.5%	6.1%
Equity Share among non Stock Options Holders	44.7%	56.0%	65.9%	76.0%
Share of Total Wealth		20.7%	7.7%	3.8%
Labor Income / Wealth	12.6%	2.9%	1.6%	0.7%

Notes: Data from SCF 1989-2013. The variable Equity Share is defined as private equity + public equity over financial wealth: (equity + bus) / net worth. Stockholders are defined as the households that hold public equity. Entrepreneurs are defined as the households with an active management role in one of the company they invest in.

could be driven by an increase in the proportion of stockholders (i.e. an extensive margin). However, Panel B of Table A1 shows that the percentage of stockholders is constant within the top percentiles (90%). The increase in the equity share is entirely driven by the increase *within* stockholders. While the heterogeneity between stockholders and non-stockholders generates a lot of variations at the bottom of the wealth distribution, these variations account for a small share of total wealth.

Investment in risky assets comes mainly in two forms: public equity and private equity. Panel A of Table A1 decomposes the increase in equity share across the top percentiles between the two types of equity. The decomposition reveals that the increase in the equity share is mostly driven by an increase in the share of wealth invested in private equity. Panel C of Table A1 shows that the proportion of entrepreneurs increases sharply in the top percentile: the proportion of households with an actively managed business is 78.5% in the Top 0.01%, compared to 10% in the general population.³⁷ Still, the share of wealth invested in equity increases both within entrepreneurs and non-entrepreneurs.

B.2 Risk Exposure Beyond the Stock Market

In the main text, I only consider the wealth exposure to equity returns. I now use data from *Forbes* to examine whether households at the top of the wealth distribution are exposed to other risk factors. To do

³⁷Similarly, Hurst and Lusardi (2004), using the Panel Study of Income Dynamics (PSID), show that the propensity of entrepreneurship increases sharply with wealth in the top percentiles.

so, I run regressions of the form

$$\sum_{s=t+1}^{t+4} \text{Within}_s - 4 \log R_{f,t+1} = \alpha_{Gh} + \sum_f \beta_f \log R_{t+1}^f + \epsilon_{G,t+1}, \quad (30)$$

where, following [Gomez \(2018\)](#), Within_{t+1} denotes the logarithmic growth between t and $t + 1$ of the average wealth of households in Forbes 400 at time t .

Column (1) in [Table A2](#) reports the results where the only factor is the excess return of the stock market. This corresponds to the specification in the last column of [Table 1](#). Column (2) reports the results for the Fama-French three factor models that add the value factor and the size factor. The exposure to the size factor is weakly negative, which suggests that households at the top tend to own bigger firms. The exposure to the value factor is not significant. Similarly, Column (3) adds the excess returns of long-term bonds as well as the excess returns of corporate bonds: only the exposure to the market is significant. Overall, the stock market appears to be the main factor for the average wealth growth of top households.

Note that the exposure of wealth in the top 400 to equity returns is relatively constant across specifications, which is reassuring, since this estimate will play a key role when calibrating the model in [Section 4](#).

Table A2: Risk Exposure Beyond the Stock Market

	Excess Average Wealth Growth in Forbes 400		
	Market (1)	FF 3-factors (2)	Market & Bonds (3)
Excess Stock Returns	0.90*** (0.26)	1.02*** (0.31)	0.89*** (0.26)
SMB		-0.12 (0.50)	
HML		0.37 (0.41)	
Long-Term Bonds			-0.07 (0.56)
Corporate Bonds			0.36 (0.64)
R^2	0.29	0.31	0.30
N	31	31	31

Notes. The table reports the results of regressing the wealth growth of top households on excess stock market returns, and a set of other factors. The left hand side is the four year cumulative “within” term. Portfolio returns of Fama-French factor models are from the Fama-French Data Library. Corporate bond returns are obtained from Ibbotson’s Stocks, Bonds, Bills and Inflation Yearbook. Estimation via OLS. Standard errors in parentheses and estimated using Newey-West with 4 lags. *, **, *** indicate significance at the 10%, 5%, 1% levels, respectively.

B.3 Alternative Measures of Top Wealth Shares

The main text measures the equity exposure at the top of the wealth distribution using wealth data from [Kopczuk and Saez \(2004\)](#) and *Forbes*. [Saez and Zucman \(2016\)](#) have recently proposed a series for top wealth shares constructed from Income Tax Returns. In [Table A3](#), I estimate the stock market exposure of top wealth percentiles using this series. Qualitatively, the results of [Section 2](#) hold true: the exposure of top wealth shares to stock market return increases with top percentile. However, the estimates are now

uniformly lower compared to [Kopczuk and Saez \(2004\)](#) or *Forbes*. For instance, the stock market exposure of the Top 0.01% is 0.66 using Income Tax Returns, compared to 0.95 using Estate Tax returns or *Forbes*.

This discrepancy comes from two distinct reasons. First, due to data limitations in income tax returns, [Saez and Zucman \(2016\)](#) must interpolate holdings across years for certain wealth categories, such as private equity. While this does not matter to capture low-frequency fluctuations in wealth inequality, this tends to artificially smooth out business cycle fluctuations³⁸ Second, as discussed in [Smith et al. \(2021\)](#), the capitalization method tends to massively overestimate the amount of bounds held by the top 0.01%, which mechanically reduces the effect of equity returns on their total wealth. For these two reasons, I only report the elasticities measured using [Kopczuk and Saez \(2004\)](#) and *Forbes* 400 in the main text.

Table A3: The Exposure to Stock Returns Across the Wealth Distribution:
[Saez and Zucman \(2016\)](#) Series

	Growth of Average Wealth in Top Percentile Groups		
	1 – 0.1%	0.1 – 0.01%	Top 0.01%
	(1)	(2)	(3)
Excess Stock Returns	0.44*** (0.10)	0.45*** (0.11)	0.65*** (0.14)
R^2	0.17	0.16	0.18
Period	1914-2007	1914-2007	1914-2007
N	94	94	94

Notes: This table reports the results of regressing the growth of the average wealth of households in a given percentile group on excess stock market returns. Data on wealth in top percentiles from [Saez and Zucman \(2016\)](#). Estimation is via OLS. Standard errors in parentheses and estimated using Newey-West with 4 lags. *, **, *** indicate significance at the 10%, 5%, 1% levels, respectively.

C Theoretical Appendix

C.1 Numerical Solution

I describe the solution of the model step by step.

Step 1. The first step of the solution method is to use the market clearing equation for equity (7) to obtain q in terms of χ :

$$p(x) = \frac{1}{x\rho_e + (1-x)\rho^\psi\chi(x)^{1-\psi}}.$$

Differentiating with respect to x gives $\partial_x p$ in terms of $\partial_x \chi$:

$$\partial_x p(x) = -p(x)^2 \left(\rho_e - \rho^\psi \chi(x)^{1-\psi} + (1-x)\rho^\psi(1-\psi)\chi(x)^{-\psi} \partial_x \chi(x) \right).$$

³⁸See <http://gabriel-zucman.eu/uswealth/> for a detailed construction of their dataset.

Differentiating a second time gives $\partial_{xx}q$ in terms of $\partial_{xx}\chi$:

$$\begin{aligned} \partial_{xx}p(x) = p(x)^2 & \left(2p(x) \left(\rho_E - \rho^\psi \chi(x)^{1-\psi} + (1-x)\rho^\psi(1-\psi)\chi(x)^{-\psi} \partial_x \chi(x) \right)^2 + 2\rho^\psi(1-\psi)\chi(x)^{-\psi} \partial_x \chi(x) \right. \\ & \left. + (1-x)\rho^\psi(1-\psi)p(x) \left(\chi(x)^{-\psi} \partial_{xx} \chi(x) - \psi \chi(x)^{-\psi-1} (\partial_x \chi(x))^2 \right) \right). \end{aligned}$$

Step 2 We then obtain the volatility of the state variable, which allows us to obtain an expression for the equity premium. Combining Ito's lemma with Lemma 1 gives an expression for $\sigma_x(x)$:

$$\sigma_x(x) = x(\alpha_E(x) - 1) \left(\sigma + \frac{\partial_x p(x)}{p(x)} \sigma_x(x) \right) \quad (31)$$

$$\implies \sigma_x(x) = \frac{x(\alpha_E(x) - 1)\sigma}{1 - x(\alpha_E(x) - 1) \frac{\partial_x p(x)}{p(x)}}. \quad (32)$$

where $\alpha_E(x) = \min\left(\alpha_E, \frac{1}{x}\right)$. Note that this equation reflects the feedback loop discussed in (20).

I then obtain the volatility of χ and p using Ito's lemma:

$$\begin{aligned} \sigma_\chi(x) &= \frac{\partial_x \chi(x)}{\chi(x)} \sigma_x(x) \\ \sigma_p(x) &= \frac{\partial_x p(x)}{p(x)} \sigma_x(x). \end{aligned}$$

One can then use the market clearing for the risky asset (8) to obtain an expression for $(\mu_R - r)(x)$:

$$\begin{aligned} 1 &= x_t \alpha_E(x) + (1-x) \left(\frac{1}{\gamma} \frac{(\mu_R - r)(x)}{\sigma_R(x)^2} + \frac{1-\gamma}{\gamma} \frac{\sigma_\chi(x)}{\sigma_R(x)} \right) \\ \implies (\mu_R - r)(x) &= \frac{1 - x\alpha_E(x)}{1-x} \gamma \sigma_R(x)^2 + (\gamma - 1) \sigma_\chi(x) \sigma_R(x) \end{aligned} \quad (33)$$

where, as given in (3), $\sigma_R(x) = \sigma + \sigma_p(x)$.

Step 3 I then obtain the drift of the state variable $\mu_x(x)$ from Lemma 1, which allows me to obtain the drift of χ and p using Ito's lemma:

$$\begin{aligned} \mu_\chi(x) &= \frac{\partial_x \chi(x)}{\chi(x)} \mu_x(x) + \frac{1}{2} \frac{\partial_{xx} \chi(x)}{\chi(x)} \sigma_x(x)^2 \\ \mu_p(x) &= \frac{\partial_x p(x)}{p(x)} \mu_x(x) + \frac{1}{2} \frac{\partial_{xx} p(x)}{p(x)} \sigma_x(x)^2. \end{aligned}$$

Finally, subtracting the expression for $(\mu_R - r)(x)$ in (33) from the expression for $\mu_R(x)$ in (3) gives an expression for $r(x)$.

Step 4 Plugging all of these quantities into the household's HJB equation (10) gives the ODE for the function χ . I solve the ODE using an accelerated finite difference method. Formally, I solve for $\chi = [\chi_1, \dots, \chi_N]$, a vector of length N corresponding to the value of the function χ on a discretized grid between 0 and 1.

Denote $F(\chi)$ the finite difference scheme corresponding to a model, where the solution satisfies $F(\chi) = 0$. I solve for χ using an iteration method. I start from an initial guess $\chi_0 = [1, \dots, 1]$ and then iterates using

the equation:

$$0 = F(\chi_{i+1}) - \frac{\chi_{i+1} - \chi_i}{\Delta}. \quad (34)$$

Each update requires solving a non-linear equation (it corresponds to a fully implicit Euler method). Economically, it is equivalent to solve for the value function today given the value function in Δ time. I solve this non-linear equation using a Newton-Raphson method. The Newton-Raphson method converges if the initial guess is close enough to the solution. Since χ_i converges towards χ_{i+1} as Δ tends to zero, one can always choose Δ low enough so that the inner steps converge. Therefore, I adjust Δ as follows. If the inner iteration does not converge, I decrease Δ . If the inner iteration converges, I increase Δ . After a few successful implicit time steps, Δ is large and, therefore the algorithm becomes like Newton-Raphson. In particular, the convergence is quadratic around the solution. I stop the iteration as soon as $F(\chi_i)$ is small enough.

This method corresponds to a method used in the fluid dynamics literature, called the Pseudo-Transient Continuation method. The algorithm with only one inner iteration and Δ constant corresponds to [Achdou et al. \(2016\)](#) (it corresponds to an semi-implicit Euler method). I find that allowing multiple inner iterations and adjusting Δ dynamically are important to ensure convergence of this non-linear PDE. This solution method is useful to solve other asset pricing models globally. It is made available in an online package <https://github.com/matthieugomez/EconPDEs.jl>

C.2 Density of Logarithmic Wealth

The density of log normalized wealth, g_t , is

$$g_t(x) = \pi g_{Et}(x) + (1 - \pi) g_{Ht}(x)$$

where g_{jt} denotes the density of normalized log wealth in group $j \in \{E, H\}$:

Applying Ito's lemma, the dynamics of log normalized wealth for individual i in group $j \in \{E, H\}$ is given by:

$$d \log w_{it} = \left(\mu_{wjt} - \frac{1}{2} \sigma_{wjt}^2 - \frac{1}{2} \nu_{wjt}^2 \right) dt + \sigma_{wjt} dZ_t + \nu_{wjt} dB_{it} \quad (35)$$

As in [Gomez \(2018\)](#), g_{jt} for $j \in \{E, H\}$ obeys a version of the Kolmogorov forward equation modified to account for aggregate shocks:

$$\begin{aligned} dg_{jt} = & \left(- \left(\mu_{wjt} - \frac{1}{2} \sigma_{wjt}^2 - \frac{1}{2} \nu_{wjt}^2 \right) \partial_x g_{jt} + \frac{1}{2} \left(\sigma_{wjt}^2 + \nu_{wjt}^2 \right) \partial_{xx} g_{jt} \right) dt - \sigma_{wjt} \partial_x g_{jt} dZ_t \\ & + (\delta + \eta) \left(1_{x_i = \log\left(\frac{\eta + \delta + \phi}{\eta + \delta}\right)} - g_{jt} \right) dt \end{aligned}$$

The following proposition gives an analytical characterization for the solution of this stochastic differential equation.

Proposition 5 (Density of log wealth).

$$g_{jt}(x) \equiv \int_{-\infty}^t (\eta + \delta) e^{-(\eta + \delta)(t-s)} \times \phi_N \left(x - \log \left(\frac{\eta + \delta + \phi}{\eta + \delta} \right) - \int_s^t \left(\left(\mu_{wju} - \frac{1}{2} \sigma_{wju}^2 - \frac{1}{2} \nu_{wju}^2 \right) du + \sigma_{wju} dZ_u \right), \int_s^t \nu_{wju}^2 du \right) ds$$

where $\phi_N(x, \sigma^2) \equiv e^{-x^2/(2\sigma^2)} / \sqrt{2\pi\sigma^2}$ denotes the density of a zero-centered normal distribution with variance σ^2 ($\phi_N(x, 0)$ should be understood as the Dirac function).

Proof. Integrating (35) over time from the time of birth s gives the following expression for log normalized wealth:

$$\log w_{it} = \log \left(\frac{\eta + \delta + \phi}{\eta + \delta} \right) + \int_s^t \left(\left(\mu_{wju} - \frac{1}{2} \sigma_{wju}^2 - \frac{1}{2} \nu_{wju}^2 \right) du + \sigma_{wju} dZ_u + \nu_{wju} dB_{iu} \right) du \quad (36)$$

This equation says that, within a cohort s and a type j , log normalized wealth follows a normal distribution with mean $\int_s^t \left(\left(\mu_{wju} - \frac{1}{2} \sigma_{wju}^2 - \frac{1}{2} \nu_{wju}^2 \right) du + \sigma_{wju} dZ_u \right)$ and variance $\int_s^t \nu_{wju}^2 du$. The overall distribution is then given by a mixture of normal distributions corresponding to each cohort in the economy.

Note that the age of the “typical” cohort present at wealth x increase with x . As x tends to infinity, households at x have been in an economy a long time, and, therefore, the wealth density of these households becomes close to the wealth density of agents in an economy with constant drift $E \left[\mu_{wju} - \frac{1}{2} \sigma_{wju}^2 - \frac{1}{2} \nu_{wju}^2 \right]$ and constant variance $E \left[\nu_{wju}^2 \right]$. This gives an intuition for the result in Proposition 2. \square

C.3 Extensions

C.3.1 Heterogeneity in Initial Endowment

In the baseline model, all agents born in the same cohort are born with the same level of initial wealth. I now discuss how the model changes when allowing the initial allocation of wealth to newborn agents to be heterogeneous across agents. More precisely, denote $m_B(\zeta)$ the ζ -th cross-sectional moment of the initial distribution of normalized wealth among newborns. As in the baseline model, imposing that the endowment of individual trees sums up to the aggregate endowment implies $m_B(1) = (\eta + \delta + \phi) / (\eta + \delta)$.

One can check that this heterogeneity in initial wealth does not change agent optimization, market clearing, or the dynamics of x_t . However, it impacts the equilibrium wealth distribution. More precisely, Lemma 1 should be modified as follows.

Lemma 4. *The law of motion of the ζ th cross-sectional moment is given by*

$$dm_{jt}(\zeta) = \left(\zeta \mu_{wjt} + \frac{1}{2} \zeta (\zeta - 1) \left(\sigma_{wjt}^2 + \nu_{wjt}^2 \right) \right) m_{jt}(\zeta) dt + \zeta \sigma_{wjt} m_{jt}(\zeta) dZ_t + (\eta + \delta) (m_B(\zeta) - m_{jt}(\zeta)) dt.$$

This is exactly the same as Lemma 2, except that the ζ -th moment of normalized wealth among newborns is now $m_b(\zeta)$ instead of $((\eta + \delta + \phi) / (\eta + \delta))^\zeta$, as in the baseline model. The proof follows the same steps as the proof of Lemma 2.

Denote ζ_B the tail index of the wealth distribution within newborn agents, i.e.

$$\zeta_B \equiv \inf\{\zeta \in \mathbb{R}_+ \cup \{+\infty\} | m_B(\zeta) = +\infty\}.$$

Proposition 2 should be modified as follows:

Proposition 6. *We have the following results:*

1. The tail index of the wealth distribution within type $j \in \{E, H\}$ is $\min(\zeta_j, \zeta_B)$.
2. The tail index of the overall distribution is $\min(\zeta_E, \zeta_H, \zeta_B)$.
3. $\min(\zeta_E, \zeta_H, \zeta_B) > 1$; that is, the wealth distribution has a right tail thinner than Zipf's law.

The proof follows the same steps as the proof of Proposition 2, with the added observation that $m_j(\zeta) = \infty$ for $\zeta \geq \zeta_B$.

Proposition 6 says that, as long as the distribution of wealth among newborn agents has a tail that is thin enough, heterogeneity in initial wealth does not matter for the tail index of the wealth distribution. This is a classical result in the literature (see, for instance, Benhabib et al., 2011 or Gabaix, 2008). This is a key reason why I focus on matching the tail index of the wealth distribution as opposed to other measures of wealth inequality when calibrating the model.

C.3.2 Labor and Capital Income

In the baseline model, individual wealth is defined as the capitalized value of the total income they will earn in their lifetime. This concept of “total” wealth differs from the observable concept of “financial” wealth, which only corresponds to the capitalized value of dividend income.

I first argue that the elasticity of “financial” wealth to stock market returns is a good proxy for the elasticity of “total” wealth to stock market returns, in the model as in the data. For a given household in the economy, denote a their financial wealth and h their human capital (i.e. the present value of their future labor income). Denote $\omega = h/(a + h)$ the ratio of human capital to total wealth.

Following the log-linearization in Campbell (1996), the growth of total wealth between two periods can be written as a weighted average of the growth of financial assets and the growth of human capital:

$$\log \frac{a' + h'}{a + h} \approx (1 - \omega) \log \frac{a'}{a} + \omega \log \frac{h'}{h}. \quad (37)$$

Projecting this approximation on stock returns gives

$$\beta_{a+h} \approx (1 - \omega)\beta_a + \omega\beta_h \quad (38)$$

This implies that the difference between the exposure of financial wealth and the exposure of total wealth to equity returns can be written as:

$$\beta_{a+h} - \beta_a \approx \omega(\beta_h - \beta_a) \quad (39)$$

This “bias” is the product of two terms: the first is the share of human capital in total wealth and the second is the difference between the exposure of human capital and financial wealth:

This equation suggests, for households at the top of the wealth distribution, the difference between the exposure of financial wealth β_a and the exposure of total wealth β_{a+h} is likely to be small as ω converges to zero as wealth converges to infinity. For instance, the IRS reports that labor income represents 8.5% of total income for top households in the U.S.³⁹ Assuming the same capitalization rate for human capital and financial assets, this suggests that human capital represents less than one tenth of financial wealth for households at the top of the wealth distribution.

Conversely, this equation also suggests that, for the average household in the economy, the difference between the exposure of financial wealth β_a and the exposure of total wealth β_{a+h} is likely to be small as $\beta_h \approx \beta_a$. This comes from the fact that, at the aggregate level, dividend and labor income are co-integrated, which implies that their permanent response to aggregate shocks must be equal.

Overall, this discussion suggests that risk exposure of the financial wealth of household at the top of the wealth distribution relative to the economy is a good approximation for the exposure their *total* wealth relative to the economy.

Moreover, the distinction between “total wealth” and “financial wealth” does not matter for the tail index of the wealth distribution either. As above, this comes from the fact that human capital accounts for such a small share of total wealth for households in the top (see Proposition 6 for a formal statement).

Overall, this discussion suggests that the distinction between total wealth and financial wealth does not matter much for the exposure of top wealth shares to stock market returns and the tail index of the wealth distribution. This is a key reason why I focus on matching these two moments when calibrating the model.

C.4 Mean Reversion in Nonlinear Models

I now propose a measure of mean-reversion for a non linear model. Denote κ the principal eigenvalue of the infinitesimal generator \mathbb{T} defined in (28)

An application of Hansen and Scheinkman (2009) for the multiplicative functional $\partial x_t / \partial x_0$ gives that, for any smooth function f , we have

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \log \partial_x \mathbb{E} [f(x_t) | x_0 = x] &= \lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E} \left[\frac{\partial x_t}{\partial x_0} \partial_x f(x_t) | x_0 = x \right] \\ &= -\kappa. \end{aligned}$$

This equality suggests that $-\kappa$ corresponds to the mean-reversion parameter for a non-linear process x_t . In particular, note that this measure of persistence does not depend on the function f or the initial value of the state variable x .

With the calibrated model, I obtain $-\kappa \approx 0.06$ which corresponds to a half-life $\log(2)/(-\kappa) \approx 11.5$ years. This is a very high value, which implies that the model gives rise to very persistent dynamic in asset prices. More precisely, it takes more than a decade for the effect of an aggregate shocks on asset valuations to decay by half (this is consistent with the impulse response functions in Figure A3).

Finally, using (29), we also get that κ also governs the convergence of $\epsilon(x, h)$ towards $\epsilon(x, \infty)$; that is,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log (\epsilon(x, h) - \epsilon(x, \infty)) = -\kappa.$$

³⁹See <https://www.irs.gov/pub/irs-soi/13intop400.pdf>.

Table A4: Elasticity of moments to parameters

	Entrepreneurs' Parameters			Households' Preferences		
	ρ_E	α_E	π	ρ	γ	ψ
Equity Premium	-0.02	-0.05	-0.01	-0.01	0.03	-0.01
STD Market Return	-0.12	0.22	-0.00	0.06	-0.17	-0.01
Average Interest Rate	0.02	0.06	0.00	0.02	-0.05	0.01
STD Interest Rate	-0.01	0.01	-0.00	0.00	-0.02	-0.00

Notes: The table reports the elasticity of each moment to each parameter. More precisely, for each moment m_i and parameter θ_j , the table reports $\frac{m_i(1.1\theta_j) - m_i(0.9\theta_j)}{0.2m_i(\theta_j)}$.

D Supplementary Figures and Tables

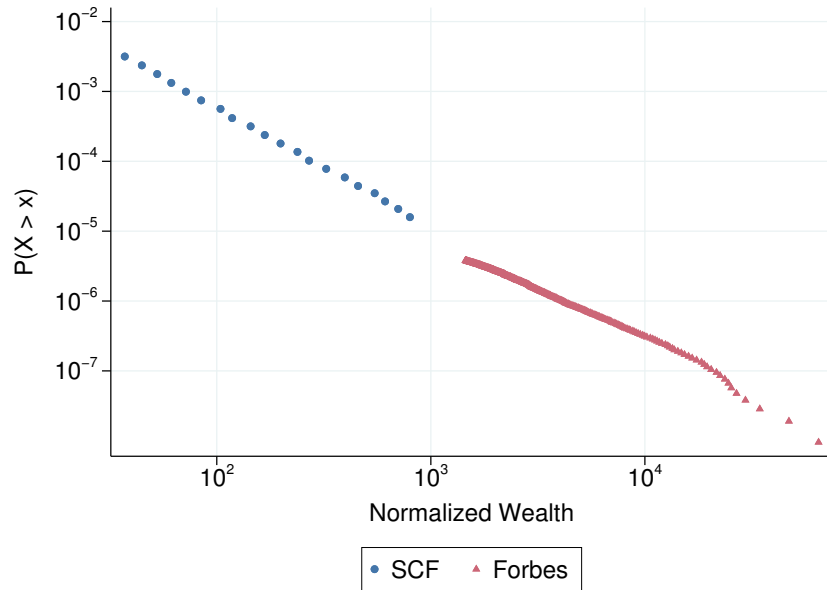


Figure A2: The Tail Index of the Wealth Distribution

Notes: This figure compares the log wealth (relative to the average wealth) to the log percentile in SCF and *Forbes*. More precisely, the figure plots the average log wealth within 40 logarithmically spaced percentile bins in SCF. The figure plots the average log wealth for each position in *Forbes* 400. The (opposite of) the slope estimate gives $\zeta = 1.5$ for SCF and for *Forbes* 400.

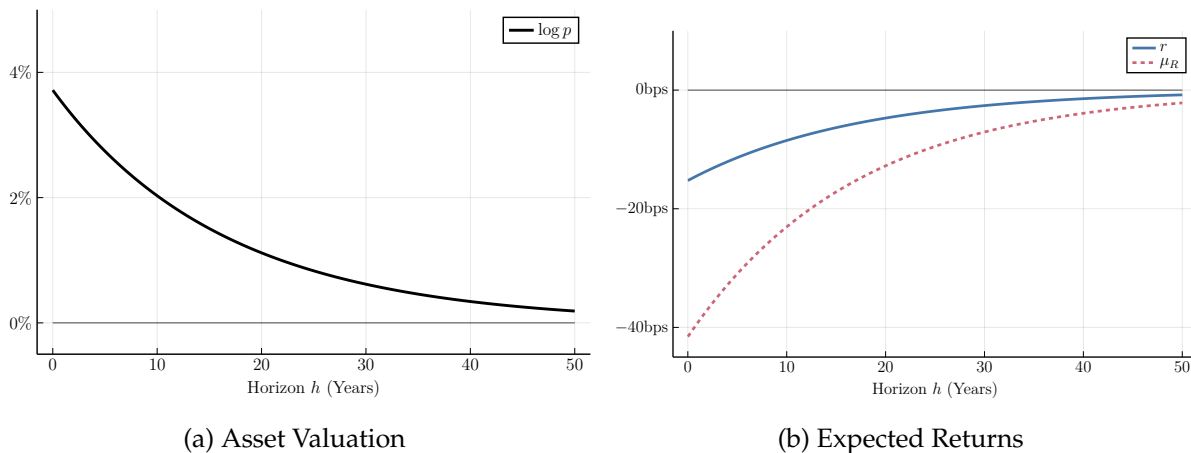


Figure A3: Impulse Response Functions

Notes: Figure A3a plots the average impulse response of $\log p_t$. Figure A3b plots the average impulse response of the risk-free rate and excess returns. The impulse response of a quantity f is defined as $\mathbb{E}[\partial_x \mathbb{E}[f(x_t)|x_0 = x] \sigma_x]$. Since the model is calibrated at the annual frequency, this can be interpreted as the (first-order) response to a one-standard-deviation annual shock in the aggregate endowment.

Table A5: The Share of Wealth Owned by the Top 0.01% And Future Excess Returns

(a) Linear Regressions

	$\log R_{M,t+1} - \log R_{f,t+1} = \alpha + \beta \log(\text{Wealth Share Top 0.01\%})_t + \gamma X_t + \epsilon_t$		
	β	γ	Adj. R^2
Log Top Share	-0.109*		3.1%
Δ Log Top Share (5 years diff.)	-0.271**		6.6%
Log Top Share + Dividend Price	-0.172**	0.116*	7.7%
Log Top Share + Dividend Payout	-0.141**	0.094	3.3%

Notes: The table reports the results of the regressions of future excess returns on the share of wealth owned by the Top 0.01% (row 1). Each row corresponds to a different set of regressors. Data on the price-dividend ratio and the price-payout ratio comes from Welch and Goyal (2008). Data on the top 0.01% wealth share comes from Kopczuk and Saez (2004). Estimation via OLS. Standard errors in parentheses and estimated using Newey-West with 4 lags. *, **, *** indicate significance at the 10%, 5%, 1% levels.

(b) Campbell and Yogo (2006) test

	Confidence Interval for $\beta \in [\underline{\beta}, \bar{\beta}]$ in the Predictability Regression (22)			
	Case with $\rho = 0.89$		Case with $\bar{\rho} = 1.02$	
	$\underline{\beta}$	$\bar{\beta}$	$\underline{\beta}$	$\bar{\beta}$
Log Top Share	-0.39	-0.07	-0.34	-0.02

Notes: The time period is 1917-1951, the longest period where the wealth share of the top 0.01% is available without missing years. This table uses the test developed by Campbell and Yogo (2006) that jointly takes into account the persistence of the predictor (the logarithmic top 0.01% wealth share), denoted by ρ , as well as its correlation with stock returns to compute the 90% confidence interval for β . The auto-regressive lag length for the DF-GLS statistic is estimated to be 1, using the Bayes information criterion (BIC).