A Q-Theory of Inequality*

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First version: March 2020
This version: January 2023
[latest version]

Abstract

We study the effect of interest rates on wealth inequality. While lower rates decrease the average growth rate of existing fortunes, we argue that they increase the growth rate of new fortunes by making it cheaper for entrepreneurs to raise capital. To understand which effect dominates, we derive a sufficient statistic for the effect of interest rates on the Pareto exponent of the wealth distribution: it depends on the lifetime equity issuance and leverage of individuals reaching the right tail of the wealth distribution. We estimate this sufficient statistic using new data on the trajectory of top fortunes in the U.S. Overall, we find that the secular decline in interest rates (or more generally the required return on wealth) has been a key contributor to the recent increase of top wealth inequality.

1 Introduction

Since the seminal contribution of Wold and Whittle (1957), a widespread view is that high interest rates tend to increase top wealth inequality. The intuition is that high rates of return increase the growth rate of existing fortunes relative to the growth rate of the economy (see Piketty and Zucman, 2015). However, this view appears to be at odds with recent data: wealth inequality has increased substantially in the past forty years in the U.S., a period marked by declining interest rates.

In this paper, we show that a lower interest rate (or more generally a lower required return on wealth) can actually increase top wealth inequality. The intuition is that, even though lower rates decrease the average growth rate of existing fortunes, they increase the growth rate of new fortunes by making it cheaper for successful entrepreneurs to raise capital.

To be concrete, consider the trajectory of successful entrepreneurs making it to the top of the wealth distribution. To finance the growth of their firms, these entrepreneurs typically raise capital from outside investors. A lower required return on wealth thus increases the growth rate of entrepreneurs, who can raise capital more cheaply. On the other hand—and as emphasized

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*We thank the following people for their comments and feedback: Jarda Borovička, Anmol Bhandari, Olivier Darmouni, Xavier Gabaix, Nicolae Gârleanu, Ben Moll, Stavros Panageas, Amir Sufi, Alexis Akira Toda, and Annette Vissing-Jorgensen. We also thank Nicolas Krakovitch and Aimé Bierdel for excellent research assistance. We acknowledge support from by the National Science Foundation under grant number SES-2117398.

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by the existing literature—lower rates decrease the growth rate of outside investors, who earn lower returns on their investments.

The overall effect of lower rates on the thickness of the right tail of wealth distribution (i.e., the Pareto exponent) depends on the type of individuals making it to the top of the wealth distribution. If, as in the U.S., individuals at the top of the wealth distribution made their fortunes as entrepreneurs who raised external financing, rather than investors, lower rates increase top wealth inequality.

We use a sufficient statistic approach to quantify the long-run effect of lower rates on the Pareto exponent of the wealth distribution. Our sufficient statistic formula depends on the lifetime equity issuance and leverage of individuals reaching the right tail of the wealth distribution. We estimate this sufficient statistic using new data on the trajectory of top fortunes in the U.S.

Our preferred estimate is that a permanent one percentage point decline in the required return on wealth generates a decline in the Pareto exponent of the wealth distribution by 4.6 log points. To put this estimate into perspective, this suggests that the 2 pp. decline in the required return on wealth that we estimate over the 1985–2015 period can account for between a third and half of the rise in Pareto inequality during this time period.\(^1\)

Overview of the paper. In Section 2, we study the effect of interest rates on wealth inequality in a stylized model with an exogenous interest rate and no aggregate risk. Entrepreneurs are born with a tree. Trees require a continuous flow of investment to grow. To finance the growth of their tree, entrepreneurs continuously sell shares to outside investors (i.e., “rentiers”). With some hazard rate, trees blossom and generate a one-time dividend equal to their size. Afterwards, entrepreneurs become rentiers themselves and invest their wealth in a diversified portfolio of trees.

In this stylized economy, we show that Pareto inequality is a u-shaped function of the interest rate. When the interest rate is sufficiently high, only rentiers make it to the right tail of the wealth distribution. In this case, lower rates decrease Pareto inequality, since it decreases the growth rate of wealth for rentiers, as in Wold and Whittle (1957) and Piketty (2015). However, when the interest rate is sufficiently low, entrepreneurs are present in the right tail of the wealth distribution. In such a case, a decline in rates increases Pareto inequality, since it increases the growth rate of wealth for entrepreneurs.

In Section 3, we derive a sufficient statistic formula for the semi-elasticity of Pareto inequality with respect to the interest rate. It depends on the equity payout yield (i.e., the net payout to equity holders over the market value of equity) and duration (i.e., the semi-elasticity of valuations with respect to the interest rate) of trees, as well as the growth rate of wealth of entrepreneurs.

We then consider a number of model extensions and describe how they affect our sufficient statistic. We first consider an extension where there is stochastic aggregate growth. In such a case, our sufficient statistic formula remains unchanged, but the correct interpretation of \( r \) is no longer “the interest rate”, but rather the required return on wealth net of per-capita

\(^1\)Pareto inequality is defined as the inverse of the Pareto exponent. A high level of Pareto inequality corresponds to a distribution with a thick right tail.
growth, a fact that will guide our measurement exercise. Then, we relax the assumption that entrepreneurs only raise external financing through equity issuance. When entrepreneurs issue both equity and debt claims, we show that another term (i.e., market leverage) enters the sufficient statistic, in order to account for the effect of debt issuance on the sensitivity of Pareto inequality to the required return. We also show that allowing investment to be optimally chosen (subject to convex adjustment costs) does not affect our sufficient statistic formula, as a result of the envelope theorem.

As a final model extension, we consider the (more realistic) case where entrepreneurs own firms that differ in their production and investment productivity, both of which evolve according to an unrestricted Markov process. Despite this arbitrary degree of heterogeneity across firms, we show that our sufficient statistic depends on the effect of required returns on the wealth trajectory of individuals making it to the top of the wealth distribution. Put differently, it depends on the effect of the required return on the past growth rate of wealth for individuals at the top of the wealth distribution, not on their current growth rate of wealth. In turn, the effect on the required return on these trajectories can be expressed in terms of the lifetime average equity payout yield and duration of the firms owned by individuals making it to the top of the wealth distribution. Intuitively, the more individuals at the top of the wealth distribution relied on external financing to grow their firm, the larger the effect of required returns on Pareto inequality.

In Section 4, we use new data on the trajectory of top fortunes to estimate these moments for the wealthiest 100 individuals in the U.S. We find that the lifetime average equity payout yield of firms owned by top individuals is around $-2.4\%$ annually, which means that entrepreneurs at the top tend to be net equity issuers throughout their lifetime. The distribution of equity payout yields is extremely skewed: some entrepreneurs own firms with a lifetime average equity payout yield as low as $-10\%$. Moreover, our data reveals an average market leverage of about 1.4 (i.e., the market value of the firm exceeds the market value of the equity). While leverage is small for firms backed by venture capital funding, it is relatively important for private firms that never raise equity over their lifetime.

Plugging these estimates into our sufficient statistic, we find that the sensitivity of Pareto inequality to the required return on wealth is large. According to our preferred measure, a 1 pp. permanent decline in the required return on wealth increases Pareto inequality by 4.6 log points. We use this estimate to quantify the contribution of declining required returns on rising Pareto inequality over the 1985–2015 period. We estimate that Pareto inequality has increased by roughly 22 pp., while the required return has declined by roughly 2 pp. A back-of-the-envelope calculation indicates that declining required returns account for between a third and half of the rise in Pareto inequality during this time period.

Finally, in Section 5, we study a general equilibrium version of our model with both capital and labor as inputs in the production process. We calibrate the model by targeting the set of micro moments that enter the sufficient statistic, in addition to important macro moments. We conduct a model experiment, which consists of feeding a sequence of MIT shocks to foreign savings to generate a permanent 2 pp. decline of the required return (i.e., a “global savings glut”). We then examine the transition dynamics of the wealth distribution. In the model, as in the data, the top 0.1% wealth share increases more than the top 1%, and the top 0.01%
increases more than the top 0.1%, which reflects a rise in Pareto inequality. The model exhibits a relatively high speed of convergence, owing to the presence of high-growth entrepreneurs who reach the top of the wealth distribution quickly.

In our model, the rise in top wealth shares can be decomposed into two terms: a capital accumulation channel (an increase in the quantity of capital owned by top entrepreneurs due to their lower cost of capital) and a revaluation channel (a relative rise in the valuation of their capital). While both channels contribute equally to the rise in the top 1% wealth share, the relative importance of the capital accumulation channel increases sharply in the right tail of the wealth distribution. This reflects the fact that top entrepreneurs disproportionately benefit from a lower cost of capital, as they raise more funds over their lifetimes.

Finally, the model allows us to examine potential general equilibrium effects that are not captured by our sufficient statistic approach. In our baseline calibration, we set the elasticity of capital to zero in order to match a constant return on capital despite falling interest rates, as we see in the data. Still, we show that the effect of lower rates on Pareto inequality persists across a wide range of calibration for the elasticity of capital.

**Related literature.** There is a large body of evidence documenting a rise in top wealth inequality in the U.S. since the 1980s (e.g., Saez and Zucman, 2016; Batty et al., 2019; Smith et al., 2022). A growing literature seeks to understand the factors behind this phenomenon. One strand of the literature focuses on the role of the return on wealth for top individuals (Piketty, 2015; Kuhn et al., 2017; Moll et al., 2022; Hubmer, Krusell and Smith Jr, 2020).² Another strand of the literature emphasizes the importance of return dispersion (Benhabib et al., 2011; Fagereng et al., 2020; Benhabib et al., 2019; Atkeson and Irie, 2020). Our findings suggest that these two factors can not be studied in isolation: we show that a decline in the average (i.e., required) return on wealth directly affects the dispersion of realized returns, which, ultimately, leads to an increase in top wealth inequality.³

Our proposed mechanism is consistent with a growing body of empirical evidence documenting the fact that the rise in top wealth inequality is driven by the rise of new fortunes, rather than the high growth rates of existing fortunes (Bach, Calvet and Sodini, 2017; Gomez, 2019; Zheng, 2019).

Our characterization of the Pareto exponent of the wealth distribution builds on the literature on random growth processes (Wold and Whittle, 1957; Jones, 2015). Recently, this literature has moved towards models with persistent growth rate heterogeneity (Luttmer, 2011; Jones and Kim, 2016; Gabaix et al., 2016). In this case, the Pareto exponent can be obtained as the principal eigenvalue of an operator related to the transition matrix between states (see de Saporta, 2005; Beare et al., 2021; Beare and Toda, 2022). Relative to that literature, a theoretical contribution of our paper is to obtain a closed-form expression for the derivative of

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²Hubmer et al. (2020) argue that the decline in tax progressivity has played a key role in increasing the average after-tax return on wealth. Kaymak and Poschke (2016) also emphasize the importance of the decline in tax progressivity.

³The assumption that idiosyncratic shocks are exogenous to interest rates is key for Piketty and Zucman (2015)’s conclusion. As they observe: “Many different kinds of individual-level random shocks play an important role in practice, and it is difficult to estimate the relative importance of each of them. One robust conclusion, however, is that for a given variance of shocks, steady-state wealth concentration is always a rising function of $r - g$.“ [emphasis added]
the Pareto exponent with respect to a parameter (here, the required return). We show that the derivative of the Pareto exponent with respect to a parameter depends on its effect on the whole wealth trajectory of individuals reaching the top of the wealth distribution. Remarkably, we show that the effect of required returns on the wealth accumulation trajectory of individuals making it to the top depends on a set of moments that can be estimated empirically. This sufficient statistic approach allows us to transparently quantify the effect of required returns on Pareto inequality in a transparent manner. Beyond the literature on top wealth inequality, several papers examine the redistributive effect of changes in the interest rate. Gărleanu and Panageas (2017), Gărleanu and Panageas (2019) and Kogan et al. (2020) build models in which lower discount rates benefit entrepreneurs at the expense of households, and Auclert (2019) studies the redistributive effect of transitory changes in the interest rate.

Our model also relates to the literature on entrepreneurial wealth accumulation (e.g., Quadrini, 2000, Cagetti and De Nardi, 2006; Moll, 2014; Guvenen et al., 2019; Peter, 2021). As in these papers, we assume that entrepreneurs remain exposed to their firms, which plays an important role in shaping the wealth distribution. One key difference is that we consider a model where firms can freely issue equity. In our model, as in the data, the most successful firms continuously raise equity and, therefore, vastly outgrow their founder. This means that lower rates matter for entrepreneur wealth through both their debt issuance and their equity issuance. In the data we find that equity issuance is most important for VC-backed firms while debt issuance is most important for private firms who never raise equity.

The most closely related paper is İmrohoroğlu and Zhao (2022), which uses a calibrated model to show that declining interest rates have contributed to the rise in wealth inequality by reducing the cost of debt for entrepreneurs. Our paper differs in that we emphasize that lower rates increase the market value of capital, which increases both the value and quantity of capital owned by top entrepreneurs. We also use tools from random growth models to obtain a closed-form formula for the effect of a lower required return on wealth on Pareto inequality, which we then map to new data on the trajectory of top U.S. entrepreneurs.

Finally, our paper relates to the large literature that seeks to understand why real interest rates have declined. Many explanations have been proposed for the secular decline in real interest rates, including a shortage of safe assets (Caballero et al., 2008), changing demographics (Carvalho et al., 2016; Auclert et al., 2021), a shift in monetary policy regime (Lettau et al., 2018), and secular stagnation (Eggertsson et al., 2019). See Mian et al. (2020) for a recent review of the evidence.

2 Stylized Model

In this section, we describe our mechanism in a stylized model of wealth inequality. Our main result is that, in the presence of entrepreneurs, Pareto inequality is a u-shaped function of the interest rate.
2.1 Environment

The economy is populated by infinitely-lived agents. Population grows at rate $\eta$. There are two types of agents: “entrepreneurs” and “rentiers”. All agents are born entrepreneurs and are endowed with a tree. Trees require outside investment to grow until they blossom. To finance the growth of their tree, entrepreneurs issue new equity shares to rentiers. When an entrepreneur’s tree blossoms, which happens only once, it produces a harvest of apples (the numéraire). The entrepreneur then becomes a rentier, who invests in a diversified portfolio of trees. We consider a small open economy equilibrium, where the interest rate $r$ is taken as given.

Trees. Each tree starts with a size of one and grows at rate $g$. To grow, the tree requires a flow of outside investment $i > 0$ proportional to its size. With constant hazard rate $\tau$, the tree blossoms and returns a one-time positive dividend equal to its size. We assume that $i < \tau$ so that each tree yields a positive cash flow in expectation. We also assume that $g < \tau + \eta$ so that the total size of trees does not grow faster than the population.

Returns. Because the cash flow of the tree is proportional to its size, the value of a tree is also proportional to its size. Denote $q$ the ratio of the value of a tree to its size. The instantaneous return of holding a tree is given by

$$
\frac{dR_t}{R_t} = \begin{cases} 
  (g - \frac{i}{q}) dt & \text{if } t < T \\
  \frac{1}{q} - 1 & \text{if } t = T 
\end{cases},
$$

where $T$ denotes the stochastic time at which the tree blossoms. While the tree is still growing (i.e., $t < T$), the return in a period $dt$ is the difference between the growth rate of the tree $g\ dt$ and the relative amount of new shares $i/q\ dt$ that must be sold to outside investors to raise $i\ dt$. This adjustment corresponds to the growth in the number of shares, or, equivalently, the extent to which existing shareholders get diluted. Finally, when the tree blossoms (i.e., $t = T$), the instantaneous return is $1/q - 1$ since the tree (with price $q$) is transformed into apples (with price 1).

Given that there is no aggregate risk, the price $q$ is pinned down by the fact that the expected return of holding a tree must equal the required return; that is, the interest rate $r$:

$$
\underbrace{r}_{\text{Required return}} = \underbrace{g - \frac{i}{q} + \tau \left(\frac{1}{q} - 1\right)}_{\text{Expected return}}.
$$

We assume $r > g - \tau$ to ensure that the price of the tree is finite, which implies that $q = (\tau - i)/(r + \tau - g)$. As usual, the price $q$ is a decreasing function of the interest rate $r$.

While a low interest rate naturally decreases the average return of holding a tree, notice that it increases the return of holding a tree conditional on it not blossoming, which is $g - i/q$. Intuitively, lower rates (i.e., higher valuations), decrease the rate at which existing shareholders
get diluted as the tree grows.

**Wealth accumulation.** Agents have log utility and discount the future at rate \( \rho \), which implies that they optimally consume a constant fraction \( \rho \) of their wealth. Our maintained assumption is that entrepreneurs must have all of their wealth invested in their tree. To finance their investment and consumption, entrepreneurs sell shares to rentiers, who hold a diversified portfolio of trees.

Let \( W_t \) be the wealth of an individual. The growth rate of wealth for an entrepreneur is their return minus the consumption rate, which gives

\[
\frac{dW_t}{W_t} = \begin{cases} 
(g - \frac{i}{q} - \rho) \ dt & \text{if } t < T \\
\frac{1}{q} - 1 & \text{if } t = T
\end{cases}
\]

(3)

where \( T \) denotes the stochastic time at which the tree blossoms.

When the tree blossoms, the entrepreneur becomes a rentier and invests in a diversified portfolio of trees. The wealth of a rentier evolves as

\[
\frac{dW_t}{W_t} = (r - \rho) \ dt.
\]

(4)

Notice that the interest rate has an opposite effect on the growth rate of wealth of entrepreneurs and rentiers. While a lower interest rate decreases the growth rate of rentiers, it increases the growth rate of successful entrepreneurs (i.e., those who own trees that keep on growing). This is shown graphically in Figure 1, which plots the total wealth of an entrepreneur with a tree that blossoms at \( T = 15 \), in a high interest rate economy as well as a low interest rate economy.

![Figure 1: Wealth trajectory of an agent who is an entrepreneur for 15 years](image)

Numerical example with \( i = 0.4 \), \( g = 0.5 \), \( \tau = 0.5 \), \( \eta = 0.05 \), \( \rho = 0.04 \)
Discussing our assumptions. We now discuss two key assumptions that we made. The first assumption is that trees require outside investment to grow (i.e., \( i > 0 \)). This assumption captures an important characteristic of young firms: they require outside funding to grow. As we will discuss in Section 4, this outside funding is a mix of equity issuance (venture capital funding or public equity offering), stock-based compensation, and debt financing. We relax this assumption in Section 3, and allow firms to endogenously have a positive or negative payout yield depending on their current production and investment productivity.

The second key assumption is that entrepreneurs must remain fully exposed to their tree. In other words, they must invest all of their wealth in their tree. This assumption captures the fact that most of the wealth of entrepreneurs is invested in their own firm (Quadrini, 2000; Cagetti and De Nardi, 2006; Roussanov, 2010). We take this as exogenous, but this type of portfolio choice constraint can result from moral hazard or asymmetric information problems (He and Krishnamurthy, 2012; Di Tella, 2017). This is a maintained assumption throughout the paper and is the core departure from the benchmark model of Wold and Whittle (1957). In Appendix A.2, we show that our model converges to Wold-Whittle when \( \tau \to \infty \), as it implies that all agents are diversified.

The key distinction between entrepreneurs and rentiers is that entrepreneurs fully invest their wealth in one tree, whereas rentiers own a diversified portfolio of trees. While our model is very stylized, the term “entrepreneur” should be understood to refer to any individual that is disproportionately exposed to a growing firm. This represents a much larger fraction of the population than strictly-defined entrepreneurs. For instance, this includes all the early employees in startups who are paid in stock-options or restricted stocks. Eisfeldt et al. (2019) reports that, in recent years, equity-based compensation accounted for 45% of total compensation to high-skilled labor in the U.S. It also includes investors with concentrated portfolios, such as venture capital investors.

2.2 Wealth distribution

We now characterize the Pareto exponent of the wealth distribution in this economy. We focus on a measure of wealth inequality (i.e., Pareto inequality) that captures the thickness of the right tail of the wealth distribution.

**Definition 1** (Pareto inequality). We say that the distribution of a random variable \( X \) has a Pareto tail if there exists a \( \zeta > 0 \) such that
\[
\lim_{x \to \infty} \frac{\log P(X > x)}{\log x} = -\zeta.
\]

The parameter \( \zeta \) is called the Pareto exponent.

A low \( \zeta \) corresponds to a thick tail (i.e., a density that decays slowly as \( x \to \infty \)). Following Jones (2015), we define Pareto inequality \( \theta \) as the inverse of the Pareto exponent (i.e., \( \theta = 1/\zeta \)). A high level of Pareto inequality thus corresponds to a thick right tail. The following proposition characterizes the steady-state level of Pareto inequality as a function of the interest rate \( r \).\(^4\)

\(^4\)To be precise, we consider a *balanced growth path* where per-capita is zero.
Proposition 2. Assume that \( \rho < g - i \). Then

\[
\theta = \max \left( \frac{g - i}{\eta + \tau}, \frac{r - \rho}{\eta} \right).
\]

The proposition says that Pareto inequality is pinned down by the maximum of two expressions. The first expression corresponds to the growth rate of successful entrepreneurs divided by the sum of population growth \( \eta \) and the Poisson rate \( \tau \) at which the tree blossoms. The second expression corresponds to the growth rate of wealth for rentiers divided by population growth. The expression for \( \theta \) is reminiscent of Proposition 4 in Luttmer (2011), who shows that the Pareto exponent of the firm size distribution is pinned down by the growth rate of high-growth firms (often referred to as “Luttmer rockets”).

Notice that the growth rate of successful entrepreneurs is decreasing in \( r \), while the growth rate of rentiers is increasing in \( r \). One can thus show that there exists \( r^* \in (g - \tau, \rho + \eta) \) such that both growth rates are equal (see proof of Proposition 2). Overall, this means that Pareto inequality is a u-shaped function of the interest rate and can be expressed as

\[
\theta = \begin{cases} 
\frac{g - i}{\eta + \tau} & \text{for } r \in (g - \tau, r^*) \\
\frac{r - \rho}{\eta} & \text{for } r \in (r^*, \rho + \eta)
\end{cases}
\]

Figure 2 plots the relationship between the interest rate and Pareto inequality in a numerical example. When \( r > r^* \) (henceforth the rentier regime), the right tail of the wealth distribution is only populated by rentiers. In this case, lower interest rates decrease Pareto inequality, since the growth rate of rentiers is increasing in the interest rate. In fact, the formula for the Pareto exponent in the rentier regime is the same as in an economy with only one type of agent (see Wold and Whittle, 1957 and Piketty and Zucman, 2015).

In contrast, when \( r < r^* \) (henceforth the entrepreneur regime), entrepreneurs are present in the right tail of the wealth distribution. As long as this is the case, lower rates increase Pareto inequality. This is because the right tail of the distribution is fully determined by the growth rate of successful entrepreneurs and those agents benefit from lower interest rates. As explained earlier, a decline in the interest rate leads to an increase of the price of a share \( q \). High valuations imply that entrepreneurs can finance the growth of their tree by issuing fewer equity shares, thereby leading to less dilution. We consider the entrepreneur regime to be the empirically relevant one. For instance, Cagetti and De Nardi, 2006 shows that most of the wealth in the top 1% of the US population is held by entrepreneurs, and, as we will see in Section 4, this is especially true at the very top of the wealth distribution.

Closing the model. For simplicity, we have treated the interest rate \( r \) as an exogenous parameter of the model. In Appendix A.3, we study a closed economy version of the stylized model, which incorporates an additional group of agents (i.e., “workers”) that always hold diversi-

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\(^5\)To be precise, the ratio of the mass of rentiers to the total mass of agents converges to one as wealth goes to infinity. See the proof of Proposition 2 in Appendix A for a formal statement.
fied portfolios. We show that, by varying the subjective discount factor of workers from zero to infinity, we can generate the range of values for \( r \) considered in Proposition 2. This justifies why we have focused on considering an exogenous interest rate faced by entrepreneurs out of parsimony.

3 Sufficient Statistic

So far, we have derived a formula for Pareto inequality in a stylized model of wealth inequality that features entrepreneurs and rentiers. We now derive a sufficient statistic formula for the long-run effect of the interest rate on Pareto inequality, \( \partial_r \log \theta \). We find that this semi-elasticity is determined by the lifetime average of the payout yield and growth rate of wealth for agents who reach the top of the wealth distribution. Section 3.1 starts by deriving the sufficient statistic in the stylized model. Section 3.2 consider a number of model extensions and shows that the sufficient statistic in the stylized model holds in a large class of models. Finally, Section 3.3 discusses transition dynamics and the importance of revaluation gains on wealth inequality, as well as the interpretation of our sufficient statistic approach in a general equilibrium context.

3.1 Sufficient statistic in the stylized model

We now derive a simple formula for the long-run effect of the interest rate on Pareto inequality in the stylized model. First, suppose that we are in the empirically relevant case, where there are entrepreneurs in the right tail of the wealth distribution (i.e., the entrepreneur regime). The semi-elasticity of Pareto inequality with respect to the interest rate:

\[
\partial_r \log \theta = \partial_r \log \left( g - \frac{i}{q} - \rho \right),
\]

\[
= -\frac{i}{q} \partial_r \log q |\frac{g - \frac{i}{q} - \rho}{g - \frac{i}{q} - \rho} + \frac{\rho}{q - \frac{i}{q} - \rho}.
\]
From the first equality, notice that the proportional change in Pareto inequality is given by the proportional change in the growth rate of wealth for entrepreneurs. The second equality shows that the change in the growth rate of wealth of entrepreneurs depends on three objects: the payout yield of the tree \(-i/q\), the sensitivity of its value to the interest rate \(|\partial_r \log q|\) (i.e., its duration), and the growth rate of wealth for successful entrepreneurs \(g - i/q - \rho\).

Equation 6 thus corresponds to a sufficient statistic for the sensitivity of Pareto inequality to \(r\). In words, we have that
\[
\partial_r \log \theta = \frac{\text{Payout yield} \times \text{Duration}}{\text{Growth rate of wealth}}. \tag{7}
\]
The sign of the payout yield pins down the sign of the sufficient statistic: as long as entrepreneurs raise external financing, then the payout yield is negative, which means that Pareto inequality is higher in a low rate environment. Recall that, in the stylized model, the payout of trees is negative when they are growing, since they require a flow of investment and do not produce anything until they blossom. Finally, the growth rate of entrepreneur wealth appears in the denominator to translate the level change in the growth rate of wealth for entrepreneurs (the numerator) into a proportional change.

While we have calculated the expression for \(\partial_r \log \theta\) under the assumption that we are in the entrepreneur regime, we show in Appendix B.1 that the sufficient statistic (7) also holds in the rentier regime, but that what matters is the payout yield and growth rate of wealth of rentiers, who hold a diversified portfolio that has a positive payout yield. This finding previews a more general result, which we establish in the next section, which is that the semi-elasticity of Pareto inequality with respect to the interest rate is determined by the lifetime average of the payout yield and growth rate of wealth for agents at the top of the wealth distribution.

### 3.2 Extensions

We now consider six extensions of the stylized model one-by-one. More precisely, we derive the sufficient statistic in six extended versions of the stylized model that incorporate, respectively: (i) stochastic aggregate growth, (ii) leverage, (iii) endogenous investment, (iv) adjustable inputs, and (v) heterogeneous firm dynamics. The key takeaway is that the sufficient statistic, or a slight modification thereof, holds over this large class of models. Of these six extensions, only the second one (leverage) substantially changes our expression for the sufficient statistic, as one needs to take into account debt issuance, not only equity issuance.

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6Consider an asset with a cash flow stream \((CF_t)_{t \geq 0}\). Using a constant required rate of return \(r\), the market value of the asset at time \(t\) is
\[
V_t = E_t \left[ \int_0^\infty e^{-r s} CF_{t+s} \, ds \right]
\]
Differentiating the (log) value of the asset with respect to \(r\), we obtain
\[
|\partial_r \log V_t| = \frac{E_t \left[ \int_0^\infty e^{-r s} CF_{t+s} \, ds \right]}{E_t \left[ \int_0^\infty e^{-r s} CF_{t+s} \, ds \right]}
\]
This equation says that, \(|\partial_r \log V_t|\) can be written as the weighted-average time to maturity of the asset’s cash flows, hence the term duration. In the stylized model, we have that \(q = (\tau - i)/(r + \tau - g)\), which means that \(|\partial \log q| = \frac{1}{\tau - i - g}\).
We now examine each model extension separately. Unless stated otherwise, all of the other assumptions from the stylized model are maintained.

**Stochastic aggregate growth.** In the stylized model, the balanced growth path features zero per-capita output growth. We now consider an extension with stochastic aggregate growth. The key takeaway is that our sufficient statistic remains unchanged, but that the symbol \( r \) should be interpreted as the required return on wealth net of per capita growth, while the symbol \( g \) should be interpreted as the growth of trees relative to per-capita growth. We now describe the environment and state the main result.

Consider the following geometric Brownian motion process for \( A_t \)

\[
\frac{dA_t}{A_t} = \gamma \, dt + \sigma \, dZ_t,
\]

where \( \gamma \) is a drift, \( \sigma \) is volatility, and \( Z_t \) is an aggregate Brownian motion. We interpret \( A_t \) as productivity, and assume that it scales per-capita output. More precisely, we assume that the each tree born at time \( t \) has an initial size \( A_t \), and that it grows by \( g \, dt + \frac{dA_t}{A_t} \) at every instant \( t \). In this case, the average size of a tree in the economy scales with \( A_t \) and \( g \) corresponds to the growth rate of the size of trees relative to the growth rate of \( A_t \).

In the stylized model, we assumed that the interest rate \( r \) was exogenous. We assume that an exogenous stochastic discount factor \( \Lambda_t \) prices all assets in the economy. We assume that it follows a geometric Brownian motion of the form

\[
\frac{d\Lambda_t}{\Lambda_t} = -rf \, dt - \kappa \, dZ_t,
\]

where \( rf \) is the risk-free rate, \( \kappa \) is the market price of risk, and \( Z_t \) is the aggregate Brownian motion.

In Appendix B.2, we show that the market pricing equation for the value of a tree in the stylized model (i.e., Equation 2) becomes

\[
rf + \kappa \sigma - \gamma = g - \frac{i}{q} + \tau \left( \frac{1}{q} - 1 \right) .
\]

where \( q \) is the expected return net of aggregate growth per capita.

Note that only aggregate risk is priced, since idiosyncratic risk (i.e., the random time at which the tree matures) can be diversified away by investors. The right-hand side is exactly as in the stylized model, which means that the formula for the market value of a tree \( q \) is the same as in the stylized model, except that \( r \) is replaced by \( rf + \kappa \sigma - \gamma \). As a result, the formula for Pareto inequality \( \theta \) is unchanged, except for the interpretation of \( r \) and \( g \) (see Appendix B.2 for a formal derivation).

Hence, under the assumption that per-capita growth does not change the growth rate of trees relative to the economy, our sufficient statistic (6) can be interpreted as the effect of a change in the required return of wealth net of per-capita growth on Pareto inequality.

\(^7\)Note that this satisfies Kaldor’s fact that the quantity of capital (i.e., the aggregate quantity of trees) grows at the same rate as per-capita output.
Leverage. So far, we have assumed that entrepreneurs finance the growth of their tree by issuing equity. We now consider a model extension where external financing takes the form of a mix of equity and debt issuance. For simplicity, we assume that (i) trees must maintain a constant ratio of debt to their size, (ii) debt has zero maturity (i.e., continuous-time equivalent of a one-period bond) with interest rate \( r_f \), and (ii) entrepreneurs hold all of their wealth in the equity of their tree. The key takeaway is that, in this case, both debt and equity issuances matter in determining the effect of lower required returns on Pareto inequality.\(^8\)

Define a tree’s book equity as its size minus outstanding debt. Let \( \lambda \) be book leverage (i.e., the ratio between size and book equity). Denote by \( i_\lambda \) the flow of investment required by equity holders as a proportion of the tree’s book equity:\(^9\)

\[
i_\lambda \equiv g - r_f + \lambda (i - (g - r_f)). \tag{8}\]

Similarly, denote \( q_\lambda \) the market value of equity divided by its book value:\(^10\)

\[
q_\lambda \equiv 1 + \lambda (q - 1), \tag{9}\]

As in the stylized model, \( q \) is pinned down by the required return on unlevered equity, \( r \). Note that we allow the interest rate on debt \( r_f \) to differ from the required return \( r \). This wedge could reflect an adjustment for aggregate risk (as in Appendix B.2), idiosyncratic risk, or, alternatively, some market segmentation between debt and equity market, as in Baker and Wurgler (2002).

The growth rate of wealth for an entrepreneur whose tree is growing is given by the equity payout yield plus the growth rate of equity minus the consumption rate:

\[
\frac{dW_t}{W_t} = \left( -\frac{i_\lambda}{q_\lambda} + g - \rho \right) dt \quad \text{if } t < T.
\]

We now consider a joint change in the risk-free rate \( dr_f \) and in the required return on unlevered equity \( dr \) on Pareto inequality (holding fixed the book leverage, \( \lambda \)). Replicating the derivation in the stylized model (see Equation 6), under the maintained assumption that we are in the entrepreneur regime, we have that\(^11\)

\[
\partial_t \log \theta = \partial_t \log \left( -\frac{i_\lambda}{q_\lambda} + g - \rho \right),
\]

\[
= dr_f + \lambda_M \left( -\frac{i_\lambda}{q_\lambda} \partial_t \log q | dr - dr_f \right) - \frac{-i_\lambda}{q_\lambda} + g - \rho, \tag{10}\]

where \( \lambda_M = \lambda q / q_\lambda \) denotes market leverage (i.e., the ratio of the market value of the tree to the

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\(^8\)We could also consider the case in which entrepreneurs own levered position in the equity of their firms. Our formulas would remain the same, after redefining \( \lambda \) to be the effective leverage of entrepreneurs.

\(^9\)The expression for \( i_\lambda \) corresponds to the total cash required by the tree per unit of book equity, \( i_\lambda \), minus the total cash given by debtholders per unit of book equity, \((g - r_f)(\lambda - 1)\).

\(^10\)The expression for \( q_\lambda \) corresponds to the market value of the tree per unit of book equity, \( q_\lambda \), minus the market value of the debt per unit of book equity, \((\lambda - 1)\).

\(^11\)We provide a detailed derivation in Appendix B.3.
market value of its equity). This equation is intuitive: a small change in \( r_f \) (holding \( r \) constant) affects Pareto inequality only if entrepreneurs are levered (i.e., \( \lambda_M \neq 1 \)) while a small change in \( r \) (holding \( r_f \) constant) affects Pareto inequality only if entrepreneurs issue equity (i.e., \( i_\lambda \neq 0 \)).

From now on, we will focus in the main text on the effect of a common change in the risk-free rate and in the required return on unlevered equity, \( \Delta r_f = \Delta r \). In this case, the semi-elasticity of Pareto inequality \( \theta \) to this common change in required returns can be written as

\[
\partial_r \log \theta = \frac{1 + \text{Market leverage} \times \text{(Equity payout yield} \times \text{Duration} - 1)}{\text{Growth rate of wealth}}.
\]  

(11)

The key takeaway is that a decline in \( r \) increases the growth rate of wealth for successful entrepreneurs for two reasons. First, as in the stylized model, a lower \( r \) increases the value of firms, which benefits entrepreneurs who issue equity. Second, in the presence of leverage, a lower \( r \) decreases interest rate payments, which benefits entrepreneurs who issue debt. In particular, even entrepreneurs who does not issue equity (as is often the case for SMEs and private firms) benefit from lower rates, as they decrease interest rate expenses.

**Endogenous investment.** In the stylized model, there is no investment decision: the amount of investment \( i \) and the growth rate of the tree \( g \) are exogenous. We now show that our sufficient statistic is robust to incorporating an endogenous investment decision.

Suppose that entrepreneurs are born with initial capital \( K \) and a technology that, given an amount of capital \( K \) allows them to (i) produce a quantity of good \( aK \) and (ii) grow capital at rate \( g \) subject to a convex adjustment cost function \( \iota(g) \). As in the stylized model, at Poisson rate \( \tau \), the capital of each firm is transformed into the consumption good one-for-one.

Denote by \( q \) the market value of a firm divided by its capital stock, which is the original definition of Tobin’s q. Given a required return \( r \), the valuation of a firm \( q \) as well are its optimal growth rate \( g \) are now pinned down by the following Hamilton Jacobi Bellman (HJB) equation:

\[
rq = \max_g \left\{ a - i(g) + gq + \tau(1 - q) \right\}.
\]  

(12)

The usual first-order condition for investment gives \( i'(g) = q \): the marginal cost of investment must equal the marginal value of capital.

In a balanced growth path, all of the model formulas from Section 2 remain unchanged, except that \( i \) is replaced by \( i(g) - a \). However, the key implication of endogenous investment is that a change in \( r \) now has the additional effect of changing the optimal growth rate of capital \( g \). Replicating the derivation in the stylized model (see Equation 6), under the maintained assumption that we are in the entrepreneur regime, we have that the effect of \( r \) on Pareto

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12 We justify this assumption in Appendix C.2.2, where we show that \( r_f \) and \( r \) appear to have declined in tandem in our sample. In the same appendix, we also quantify separately the effect of a change in the required return on unlevered equity \( \Delta r_f \) and the effect of a change in the interest on corporate debt \( \Delta r_f \) on Pareto inequality.

13 The stylized model can be seen as special case with no production and infinite adjustment costs; that is, \( a = 0 \), and \( i(\phi) = i \) if \( \phi = g \) and \( +\infty \) otherwise.
inequality is given by

\[
\partial_r \log \theta = \partial_r \log \left( \frac{a - i(g)}{q} + g - \rho \right),
\]

\[
= \frac{a - i(g)}{q} \partial_r \log q + \frac{1 - i'(g)}{q} - \partial_r g. \tag{Baseline sufficient statistic}
\]

The first term on the right-hand side maps to the data exactly in the same way as in the stylized model (i.e., payout yield times duration divided by growth rate of wealth). The second term, which accounts for the response of investment to \(r\), is new. However, given that the optimal growth rate \(g\) ensures that the entrepreneur invests up to the point where the marginal cost of investment \(i'(g)\) equals its marginal value \(q\), the second term is zero (i.e., an application of the envelope theorem). The key takeaway is that, at the first order, the response of investment to lower required returns does not matter for the sensitivity of Pareto inequality to required returns. In particular, in the case where investment reacts a lot to changes in required returns, entrepreneurs simply end up with a smaller fraction of larger firms.\(^{14}\)

Note that this result relies on the fact that investment is optimally chosen (i.e., \(i'(g) = g\)).

In Appendix B.4, we consider a model extension where investment decisions are distorted due to a financial constraint on the equity payout yield. The key takeaway is that the presence of binding financial constraints amplifies the sensitivity of Pareto inequality to the required return. In our empirical application, we focus on U.S. entrepreneurs reaching the right tail of the wealth distribution. For these entrepreneurs, financial frictions may not be that important. When we bring the theory to the data, we do not quantify the amplification effect of financial constraints. This is a conservative choice: if successful entrepreneurs are indeed financially constrained, then our results underestimate the effect of \(r\) on Pareto inequality.

**Adjustable input.** In the model with endogenous investment, we assumed that the production function only used capital as input. However, our model can easily be generalized to multiple inputs. To be more concrete, suppose that the production function is \(F(K, L)\), where \(K\) denotes any fixed factor limiting the growth of the firm (i.e., the one subject to adjustment friction) and \(L\) denotes a perfectly adjustable factor (or a combination of them). We assume that \(F\) is homogeneous of degree one, which allows us to write \(F(K, L) = \tilde{F}(l)\), where \(l \equiv L/K\) is the intensity of the adjustable relative to the fixed factor and \(\tilde{F}(l) \equiv F(1,l)\).

With such a production function, the entrepreneur problem is to choose an intensity of the adjustable factor \(l\) and a growth rate of the fixed factor \(g\), as to maximize the value of the firm

\[
rq = \max_{l, g} \{ f(l) -wl -i(g) + gq + \tau(1-q) \},
\]

where \(w\) denotes price of the adjustable input. This gives the same equation as the HJB (12) in the Endogenous investment extension after redefining \(a = \max f(l) -wl\). In particular,

\(^{14}\) For the same reason, the size of the investment response does not matter for the elasticity of the firm value to the interest rate, \(\partial_r \log q\), which can still be inferred from the maturity of its cash flows (see Footnote 6).
we obtain the same expression for the sufficient statistic—the only difference is that the equity payout yield in this context is now \((f(l) - w_l - \iota(g))/q\).

There are two takeaways from this generalization. First, what we call capital should be understood as any fixed factor in the production function of the entrepreneur: it could refer to tangible capital (Hayashi, 1982) or to some form of intangible capital, such as knowledge capital (Klette and Kortum, 2004, Luttmer, 2011), organization capital (Lucas Jr, 1978) or the firm’s customer base (Gourio and Rudanko, 2014, Bhandari and McGrattan, 2021). Similarly, our notion of adjustment cost should be understood as any monetary cost paid by the entrepreneur to build this capital over time. Second, in the presence of an adjustable input, the market price \(w\) now enters (negatively) in the cash flow that the firm pays out to firm owners. As we discuss below in Section 3.3, this matters for the interpretation of our sufficient statistic in general equilibrium.

**Heterogeneous firm dynamics.** In the stylized model, the life cycle of a firm is broken down into an investment state (i.e., when the tree grows) followed by a production state (i.e., when the tree blossoms). We made this simplifying assumption because it implies simple analytical expression for Pareto inequality \(\theta\). We now show that the key insights from the stylized model hold in an general model of firm dynamics, where firms transition stochastically between different states, in each of which there can be both production and investment happening.

We assume that firm heterogeneity is fully summarized by a state \(s \in \{1, \ldots, S\}\), which follows a continuous-time finite-state Markov chain with transition rate matrix \(T\).\(^{15}\) Agents are born entrepreneur and their firm has initial size \(K\) and initial state drawn from an arbitrary distribution \(\psi\). At Poisson rate \(\tau\), entrepreneurs diversify and become rentiers.\(^{16}\) As in the stylized model, we assume that both the production and investment technology features constant returns to scale, which implies that the value of a firm with capital stock \(K\) in state \(s\) is given by \(q_s K\), where \((q_s)_{1 \leq s \leq S}\) is the solution to

\[
 rq_s = \max_s \left\{ a_s - \iota_s(g) + g q_s + (T q)_s \right\}. \tag{13}
\]

As in the stylized model, the annuity value of a firm \(rq_s\) is determined by the current payout, the growth rate, and the expected contribution of idiosyncratic shocks. Unlike in the stylized model, we do not make any assumption on the sign of the payout \(a_s - \iota_s(g)\). Going forward, we assume that there exists a unique, strictly positive solution to (13).\(^{17}\)

The parameter \(a_s\) denotes the gross return on capital, which is a measure of production efficiency. The term \(\iota_s(g)\) denotes the investment rate. Notice that we allow the adjustment cost function \(\iota_s(\cdot)\) to depend on the state \(s\), which captures differences in investment efficiency.

\(^{15}\)More precisely, \(T\) is a \(S \times S\) matrix: its off-diagonal elements \(T_{ss'}\) contain the Poisson rates at which firms transition from state \(s\) to state \(s'\) for \(s \neq s'\). Its diagonal elements \(T_{ss}\) correspond to \(-\sum_{s'=s} T_{ss'}\). In particular, note that we have \(E_t [df(s_t)|s_t=s] = (T f)(s) dt\) for any function \(f\) defined on the set of states \(\{1, \ldots, S\}\).

\(^{16}\)In the stylized model, the parameter \(\tau\) governs both the Poisson rate at which the tree blossoms and the Poisson rate at which entrepreneurs diversify. We now relax this assumption and allow the diversification event to be independent from the state of the firm.

\(^{17}\)This is analogous to our assumptions in the stylized models that (i) trees have a positive payout in expectation \((i < \tau)\) and (ii) the value of a tree is finite \((i.e., \tau > g - \tau)\).
For instance, a state could be associated with low production efficiency and a high investment efficiency, in which case the payout would be negative (i.e., the firm raises equity). Another state could be associated with a high production efficiency and low production efficiency, in which case the payout would be positive (i.e., the firm pays dividends).

The following proposition characterizes Pareto inequality $\theta$ along a balanced growth path, which can be expressed in terms of the vector $\mu = (\mu_1, \ldots, \mu_S)$, where $\mu_s \equiv \frac{a_s - \eta_s(g_s)}{q_s} + g_s - \rho$ is the growth rate of the capital owned by an entrepreneur in state $s$.

**Proposition 3** (Pareto tail). Suppose that there is at least one productivity state $s$ such that $\mu_s > 0$. Then, the distribution of wealth has a Pareto tail with Pareto inequality given by

$$\theta = \max \left( \theta_E, \theta_R \right),$$

where $\theta_R = \frac{r - \rho}{\eta}$ and $\theta_E$ is the unique positive number such that

$$\rho_D \left( \frac{1}{\theta_E}D(\mu) + T - (\eta + \tau)I \right) = 0,$$

where $\rho_D(\cdot)$ denotes the dominant eigenvalue of a matrix, $D(\mu)$ denotes the diagonal matrix with elements $\mu$ on its main diagonal, and $I$ denotes the identity matrix.

Notice that Proposition 3 is a strict generalization of Proposition 2. Indeed, in the stylized model, we have that $T = 0$ (there is only one state) and $D(\mu) = g - \frac{1}{q} - \rho$. Plugging these expressions in (14), we recover the result from Proposition 2.

The following proposition characterizes the object $\partial_r \log \theta$ under the maintained assumption that we are in the entrepreneur regime (or equivalently $\theta_E > \theta_R$).

**Proposition 4** (Sufficient statistic). The semi-elasticity of Pareto inequality $\theta$ to the required return $r$ along a balanced growth path is given by

$$\partial_r \log \theta = \lim_{W \to +\infty} \mathbb{E} \left[ \frac{1}{\tau_i \int_0^{\tau_i} \partial_r \mu_{s_{it}} \, dt}{\frac{1}{\tau_i \int_0^{\tau_i} \mu_{s_{it}} \, dt}} \left| W_i = W \right. \right],$$

where $s_{it}$ denotes the state of individual $i$ at age $t$, while $W_i$ and $\tau_i$ denote respectively the current wealth and age of an individual $i$.

The key takeaway is that the relative effect of a change in the required return on Pareto inequality $\theta$ is given its relative effect on the lifetime average rate of capital accumulation of individuals at the top of the wealth distribution. Using the fact that $\partial_r \mu_s = \frac{a_s - \eta_s(g_s)}{q_s} \left| \partial_r \log q_s \right|$, we obtain that the following sufficient statistic in words:

$$\partial_r \log \theta = \mathbb{E} \left[ \frac{\text{Lifetime average of Payout yield \times Duration}}{\text{Lifetime average of Growth rate of wealth}} \right| \text{Being in top percentile} \right].$$

As in the stylized model, the sufficient statistic depends on the payout yield, duration, and growth rate of wealth. However, what matters for the sensitivity of Pareto inequality to the required return is the lifetime average payout yield, duration, and growth rate of wealth conditional on reaching the top of the wealth distribution.
Note that Proposition 4 is a strict generalization of the sufficient statistic in the stylized model, in which there was no return heterogeneity between entrepreneurs. The insight from Proposition 4 is key to our measurement exercise in Section 4, where we measure backward looking lifetime averages for individual currently at the top of the wealth distribution.

3.3 Discussions

We now discuss three topics that are important to understand the effect of a change in the required return $r$ on wealth inequality: general equilibrium considerations, the difference between the return on wealth and the return on capital, and revaluation gains.

General equilibrium. So far, our thought experiment has been to consider a partial equilibrium change in the required return $r$. A strength of this approach is that it allows us to remain agnostic with regard to the exact source of the change in $r$. We now discuss how to interpret our sufficient statistic approach when the required return $r$ is determined in general equilibrium (i.e., in a closed economy).

First, the change in the equilibrium interest rate could come from a shift in asset-demand that originates outside of the entrepreneurial sector. In this case, as long as this shift does not affect Pareto inequality directly, our sufficient statistic captures the total effect of this shift on Pareto inequality. This is the situation described in Appendix A.3, where we show that a change in the subjective discount factor of “workers” generates a change in the equilibrium required return on wealth, without changing Pareto inequality directly.

In general, however, our sufficient statistic only captures the change in Pareto inequality that is attributable to the change in the required return. To give a concrete example, consider the Adjustable input model extension, where the adjustable input is labor. Suppose that there is a decline in the subjective discount factor of entrepreneurs $\rho$ (i.e., a rise of their savings rate), which leads to a decline of the required return $r$ declines and a rise of wages $w$. In this example, the total change in Pareto inequality can thus be expressed as

$$
\text{d log } \theta = \partial_\rho \text{ log } \theta \times \text{d}\rho + \partial_r \text{ log } \theta \times \text{d}r + \partial_w \text{ log } \theta \times \text{d}w.
$$

The first term accounts for the direct effect of the change in $\rho$, which is the rise in Pareto inequality due to the higher savings rate of entrepreneurs. The second term accounts for the decline in $r$, which affects Pareto inequality via the change in the cost of capital faced by entrepreneurs, as stressed in this paper. Finally, the third term accounts for the rise of wages $w$, which affects Pareto inequality by reducing the cash flows distributed to the owners of the firm.

Our sufficient statistic approach thus allows us to transparently isolate the contribution of changes in $r$ on Pareto inequality, without taking a stance on the reason what the underlying shock is declined. In Section 5, we conduct a general equilibrium model experiment to quantify the different equilibrium effects jointly.
Return on capital versus return on wealth. In the neoclassical growth model, the equilibrium interest rate is equal to the marginal return on capital. We now define the return on capital in our model and explain why, in general, it is not equal to the return on wealth. Using the notation from the Endogenous investment extension, the definition of the net return on capital is

\[ \text{rok} \equiv a + (g - \iota(g)). \] (17)

The net return on capital is the sum of production efficiency \( a \) (i.e., how much gross output is produced per unit of capital) and investment efficiency \( g - \iota(g) \) (i.e., the difference between the growth rate of capital and the investment rate). This definition of the net return on capital fully summarizes the technological contribution of a firm to aggregate net value-added, and is consistent with the System of National Accounts.\(^{18}\)

Plugging this definition into the expression for \( q \) given in (12), we can write:

\[ q = 1 + \frac{\text{rok} - r}{r + \tau - g}. \] (18)

The second term in (18) represents the present value of rents. Indeed, from the perspective of a firm owner, the average return on investment is rok while the marginal return on investment is \( r \), which means that investment generates Ricardian rents that accrue to firm owners.\(^{19}\) Note that the presence of a convex investment adjustment cost function allows \( \text{rok} > r \) to be sustained in equilibrium. The reason is that convex adjustment costs generate decreasing returns to scale in investment. Hence, rents accrue to firm owners due to the fact that their technology cannot be costlessly scaled to accommodate the supply of savings. We make this point quantitatively in Section 5, where we simulate a rise in savings in a general equilibrium model.

Note that we can write the return on wealth for entrepreneurs, while their firms keep growing, as

\[ \frac{dR_t}{R_t} = \text{rok} \, dt + (g - \text{rok}) \left(1 - \frac{1}{q}\right) \, dt. \] (19)

When entrepreneurs do not use external financing (i.e., \( g = \text{rok} \), or equivalently \( a = \iota(g) \)), their return on wealth is equal to the return on capital. However, when they use external

\(^{18}\)The net return on capital is defined as net capital income over capital. In the National Accounts, net capital income is the sum of gross profits minus capital depreciation. Using our notation, gross profits are \( aK \) and the identity that implicitly defines the depreciation rate is

\[ K_{t+1} - K_t = -\text{depreciation rate} \times K_t + I_t, \]

where \( K_t \) is capital and \( I_t \) is investment. Hence, the “depreciation rate” in our model is \( \iota(g) - g \). Putting together, we have that the net return on capital in our model is \( a - (\iota(g) - g) \), which coincides with (17).

\(^{19}\)In Appendix B.6, we show that this fact (i.e., Tobin’s \( q \) is equal to one plus the present value of rents) holds in a much more general environment. Several recent papers use the idea that, at the aggregate level, \( (\text{rok} - r)K \) is the total value of “pure profits” that accrue to firm owners (e.g., Barkai, 2020; Karabarbounis and Neiman, 2019, and Gouin-Bonenfant, 2022). In our model, the pure profits are due to Ricardian rents, but, in general, they can be due to market power. See Cochrane (1991) for a formal proof that the first order condition for investment in q-theory models (i.e., \( \iota'(g) = q \) using our notation) implies that the marginal return on investment is equal to the discount rate (i.e., \( r \) using our notation).
financing (i.e., \( g > r_k \), or equivalently \( a < \iota(g) \)), their return on wealth differs from the return on capital. In particular, their return on wealth exceeds the return on capital whenever \( q \geq 1 \). This comes from the fact that part of an entrepreneur’s return comes from selling shares to outsiders. This is the key insight of our paper: for a given return on capital, a low cost of capital \( r \) benefits entrepreneurs who raise external financing.

**Revaluation gains.** For now, we have focused on how the level of the required return on wealth \( r \) affects the rate of capital accumulation of each individual, and, therefore, the distribution of capital in the economy. At the same time, a decline in required returns also increases the market value of this capital. If this revaluation effect is heterogeneous across households, this may shift the observed distribution of wealth.

To fix ideas, we examine these two effects in the context of the stylized model. Integrating the law of motion for wealth (3) over time gives us the wealth level of an entrepreneur born \( t \) periods ago

\[
W_t = \text{Tree valuation} \times e^{(g - \frac{i}{t}\rho)t},
\]

It is the product of the valuation of the tree \( q \) times the quantity of tree that they own. The point that we have made so far is that a lower required return \( r \) increases the quantity of tree owned by an entrepreneur by reducing the dilution rate (a “capital accumulation” channel). An additional effect of lower required returns is that they increase the valuation of trees that they own (a “capital revaluation” channel).

In the stylized model, there is a single asset, which means that all agents experience the same proportional revaluation of their wealth. Hence, the revaluation channel does not affect wealth inequality. In a more general model with multiple types of assets, however, revaluation gains can be heterogeneous across agents, which would affect wealth inequality. In particular, the revaluation channel could increase top wealth shares if agents at the top of the wealth distribution tend to have levered positions in firms (as in the Leverage extension) or if they tend to own firms with a higher duration than the rest of the economy (as in the Heterogeneous firm dynamics extension).

One important distinction between the accumulation and revaluation channels, however, is that only the capital accumulation channel affects Pareto inequality.\(^{21}\) This comes from the fact that the revaluation channel only affects the level of wealth of an entrepreneur while the capital accumulation channel affects its growth rate. In contrast with the revaluation channel, the capital accumulation channel disproportionately affects the most successful entrepreneurs—that is because a lower cost of capital disproportionately favors those who go through a high number of funding rounds. As such, while the revaluation channel tend to affect all top wealth shares equally, the contribution of the capital accumulation channel increases in the right tail

\(^{20}\)See Gomez (2016) for empirical evidence on heterogeneous leverage across the wealth distribution and Greenwald et al. (2021) for empirical evidence on asset durations across the wealth distribution.

\(^{21}\)Formally, in any model in which the wealth of individual \( i \) is given by \( q_i K_i \), where \( K_i \) denotes the quantity of capital that they own and \( q_i \) is the valuation of this capital (a bounded random variable), we have that Pareto inequality in wealth is the same as Pareto inequality in capital (see Gabaix, 2016).
of the wealth distribution. In Section 5, we will quantify the relative effect of each channel for the wealth share of the top 1%, 0.1%, and 0.01% in a calibrated model disciplined by existing empirical evidence on leverage and duration across the wealth distribution.

4 Empirics

In this section, we use data from various sources to estimate our sufficient statistic for the effect of required returns \( r \) on Pareto inequality \( \theta \). Given a reference year \( T \) and a sample of \( i = 1, \ldots, N \) individuals currently in the right tail of the wealth distribution, we estimate our sufficient statistic with

\[
\frac{1}{N} \sum_{i=1}^{N} \left( 1 + \text{Market leverage}_{i,T} \times \left( \text{Equity payout yield}_{i,T} \times \text{Duration} - 1 \right) \right)
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} \text{Growth rate}_{i,T},
\]

(21)

where Equity payout yield\(_{i,T} \), Market leverage\(_{i,T} \), and Growth rate\(_{i,T} \) are lifetime averages for each individual \( i \) in the top percentile of the wealth distribution in \( T \). Equation 21 thus corresponds to the empirical analogue of the sufficient statistic in the stylized model (see Equation 7) augmented to account for leverage (see Equation 11) and heterogeneous firm dynamics (see Equation 15).\(^{22}\) As we will discuss shortly, we do not attempt to estimate firm duration at the individual level, but instead treat it as a parameter to be estimated.

While the existing literature focuses on the characteristics of individuals at the top of the wealth distribution (e.g., Cagetti and De Nardi, 2006), relatively little is known regarding the trajectory of individuals reaching the top of the wealth distribution. Hence, a contribution of our paper is to construct a database on the growth rate of wealth, equity payout yield, and leverage of individuals reaching the top of the wealth of the wealth distribution.

4.1 Estimating the sufficient statistic

**Forbes list.** We identify individuals in the right tail of the wealth distribution using the list of the wealthiest 400 Americans produced by Forbes Magazine. The list is created by the staff of the magazine based on a mix of public and private information.\(^{23}\) For our application, we choose 2015 as the reference year and define the “right tail” as individuals in the top 100, a group for which information is widely available.

Table 1 contains information on the top 100 individuals included in the Forbes list in 2015. We assign to each individual the main firm that they or their family founded. Out of this set

\(^{22}\)Note that, relative to Equation 15, we calculate a ratio of averages, instead of an average of ratios. We make this choice in order to make the calibration strategy in Section 5—where we calibrate our model by targeting each of the individual moments (market leverage, equity payout yield, and growth rate of wealth) separately—entirely comparable to the sufficient statistic approach that we describe in this section. In Appendix C.1.2, we use estimate an alternative estimator (i.e., the average of ratios) and show that we obtain very similar results.

\(^{23}\)Forbes Magazine reports: “We pored over hundreds of Securities Exchange Commission documents, court records, probate records, federal financial disclosures and Web and print stories. We took into account all assets: stakes in public and private companies, real estate, art, yachts, planes, ranches, vineyards, jewelry, car collections and more. We also factored in debt. Of course, we don’t pretend to know what is listed on each billionaire’s private balance sheet, although some candidates do provide paperwork to that effect.”
Table 1: Individuals in the top 100 (Forbes list, 2015)

<table>
<thead>
<tr>
<th>Group</th>
<th>Count</th>
</tr>
</thead>
<tbody>
<tr>
<td>Entrepreneurs</td>
<td>71</td>
</tr>
<tr>
<td>Public corporation</td>
<td>41</td>
</tr>
<tr>
<td>Private corporation</td>
<td>30</td>
</tr>
<tr>
<td>Rentiers</td>
<td>4</td>
</tr>
<tr>
<td>Financiers</td>
<td>25</td>
</tr>
</tbody>
</table>

Notes. “Entrepreneurs” are defined as individuals who are invested in non-financial firms that they (or a family member) founded; “Rentiers” are defined as individuals who are no longer invested in the firm that they (or a family member) founded; “Financiers” are defined as individuals who are invested in a financial firm that they (or a family member) founded. Data are from Forbes.

of individuals, we remove 4 “rentiers”, which we define as individuals who are no longer invested in the firm that they or their family founded. As discussed in the context of the stylized model, in the entrepreneur regime, the Pareto exponent of the wealth distribution is entirely determined by the growth rate of wealth of entrepreneurs (see Section 2). We also remove 25 “financiers”, which we define as individuals who own a financial firm, since our framework does not directly apply to them. For a thorough investigation of the importance of financiers at the top of the income distribution, see Kaplan and Rauh (2010). We are left with 71 individuals for which we have detailed information (age, wealth, source of wealth, firms that they founded, etc.). As reported in Table 1, roughly 60% own public firms while the rest own private firms.

Equity payout yield. The equity payout yield is defined as the total cash flows distributed to equity holders over a given period of time (in our case a year), divided by the market value of this equity. These cash flows can be distributed either through dividends or share repurchases (i.e., buybacks). As a consequence, the equity payout yield can be written as sum of the dividend yield (i.e., dividends distributed divided by the market value of the firm equity) and the buyback yield (i.e., cash flows distributed through shares repurchases minus the cash flows received by the firm through share issuances, divided by the market value of the firm equity). We now describe our methodology (more details are given in Appendix C.1.1).

We start by computing the lifetime average dividend yield of each firm in our sample. In years for which the firm is public, we compute its dividend yield as the ratio of dividends to the market value of their equity, using data from Compustat. In years for which they are private, we set their dividend yield to zero (we justify this assumption shortly).

We now turn to the lifetime average buyback yield. Remember that, by definition, the buyback yield is equal to the cash flows distributed by the firm through share repurchases minus the cash flows received through share issuances, divided by the market value of the firm. Put differently, the numerator is the (net) number of shares sold in this year times the price of each share, while the denominator is the number of shares outstanding times the price of each share. As a result, the buyback yield in a given year can be measured as the growth rate of the number of shares outstanding during that year (adjusted for stock split). This implies that we can directly estimate the lifetime average buyback yield of a firm as the growth of its
number of shares since the founding date:

\[
\text{Lifetime average buyback yield} = -\log\left(\frac{N_{2015}}{N_{t_0}}\right) \frac{2015-t_0}{N_{2015}}, \tag{22}
\]

where \(N_{2015}\) denotes the number of shares in 2015 and \(N_{t_0}\) denotes the number of shares when the firm was founded. An important advantage of this method is that we do not need to know the total amount of equity raised through the multiple funding rounds, or even the implied valuation at each round, which is not always publicly available.

For firms that are public in 2015, we measure \(N_{2015}\) as the number of common and reserved shares in 2015 (adjusted for any stock split since the IPO) and \(N_{t_0}\) as the number of shares owned by all founders at the time of the IPO (reported on the S-1 filing).\(^{24}\)

For firms that are private in 2015, we set \(N_{2015} = N_{t_0}\), which effectively sets their lifetime average buyback yield to zero. This reflects the fact that, in our sample, these firms do not appear to have issued or repurchased shares over their lifetime. As explained above, for these “always private” firms, we also set the dividend yield to zero, which implies that their lifetime equity payout yield is zero. This methodological choice effectively shuts down the contribution of net equity issuance in the sufficient statistic (21): for the entrepreneurs owning these firms, only their leverage matters in determining how much they benefited from lower required returns.

We report summary statistics for the lifetime average dividend yield and buyback yield of each firm in our sample in Table 2. We find that firms owned by individuals in the right tail of the wealth distribution in 2015 have had a lifetime average dividend yield of 0.6% and a lifetime average buyback yield of −2.9%. Overall, their average equity payout yield has been −2.4% since their founding. The key observation here is that these firms, on net, have raised more cash from equity holders than they have distributed back to them.\(^{25}\) Notice that the distribution of the (lifetime average) equity payout yield is highly negatively skewed, with values ranging from −17.7% for Travis Kalanick (Uber) to 2.4% for Les Wexner (L brands).

### Table 2: Summary statistics

<table>
<thead>
<tr>
<th>Obs.</th>
<th>Average</th>
<th>Percentiles</th>
<th>Min</th>
<th>p25</th>
<th>p50</th>
<th>p75</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>71</td>
<td>−2.4%</td>
<td>−17.7%</td>
<td>−2.2%</td>
<td>0.0%</td>
<td>0.0%</td>
<td>2.4%</td>
<td></td>
</tr>
<tr>
<td>71</td>
<td>0.6%</td>
<td>0.0%</td>
<td>0.0%</td>
<td>0.0%</td>
<td>1.0%</td>
<td>6.7%</td>
<td></td>
</tr>
<tr>
<td>71</td>
<td>−2.9%</td>
<td>−17.7%</td>
<td>−4.6%</td>
<td>−0.6%</td>
<td>0.0%</td>
<td>1.2%</td>
<td></td>
</tr>
<tr>
<td>71</td>
<td>1.37</td>
<td>1.00</td>
<td>1.13</td>
<td>1.37</td>
<td>1.37</td>
<td>3.89</td>
<td></td>
</tr>
<tr>
<td>71</td>
<td>0.31</td>
<td>0.06</td>
<td>0.16</td>
<td>0.23</td>
<td>0.38</td>
<td>1.33</td>
<td></td>
</tr>
</tbody>
</table>

**Notes.** This table reports the lifetime average dividend yield, buyback yield, market leverage, and growth rate of the top 100 U.S. individuals in 2015. The construction of each variable is detailed in Appendix C.1.1. Data are from Forbes, Compustat, and SEC S-1 filings.

\(^{24}\)Indeed, this is equal to the total number of shares held by founders when the firm was launched, assuming that founders neither sell or receive shares before the IPO. See Appendix C.1.1 for more details.

\(^{25}\)Note that this implies that these firms are not representative of the corporate sector as a whole, which always has a positive payout yield. See Abel et al. (1989) for a theoretical argument for why dynamic efficiency implies a positive aggregate payout yield, as well as empirical evidence.
**Market leverage.** We estimate the market leverage as follows. For firms who were public in 2015, we compute the average market leverage for every post-IPO years using data from Compustat (see Appendix C.1.1). We find an average market leverage of 1.37. We use this figure to impute the market leverage of firms that were private in 2015.

**Growth rate of wealth.** We estimate the lifetime average growth rate of wealth for each individual as the log ratio between wealth in 2015 and initial wealth, divided by the age of the firm. Formally, we use the formula

\[
\text{Lifetime average growth rate} = \frac{\log (W_{2015}/W_{t_0})}{2015 - t_0}, \tag{23}
\]

where \(W_t\) denotes the wealth of an individual at time \(t\) normalized by the average wealth in the economy, and \(t_0\) denotes the founding date of the firm.\(^{26}\) Unfortunately, there is very limited evidence on the wealth of our entrepreneurs at founding date, so, as a baseline, we set \(W_{t_0} = 1\) (i.e., we assume that their initial wealth equals the average wealth in the economy). In Appendix C.1.2, we show that our sufficient statistic does not change much when we set \(W_{t_0} = 1/5\) or \(W_{t_0} = 5\); this is because the terminal wealth of our individuals, \(W_{2015}\), is typically several orders of magnitude larger than the average wealth in the economy, and so their exact starting point does not matter much.

As shown in Table 2, we estimate an average growth rate of 31%. The distribution of growth rates is positively skewed, with large outliers corresponding to Facebook and Uber founders. In contrast, heirs of entrepreneurs who founded firms in the distant past tend to have much lower average growth rates.

**Duration.** The effect of the required return on Pareto inequality depends on the average duration of the firms owned by individuals that reach the top of the wealth distribution, where duration is defined as the semi-elasticity of a firm’s market value with respect to the required rate of return, in absolute value.

Ideally, we would measure firm duration as the reaction of a firm’s market value to an unexpected and permanent change in the required return on wealth. However, this is hard to do empirically. In particular, unexpected monetary policy shocks correspond mostly to transitory changes in short-term interest rates. For this reason, we do not attempt to measure duration for each firm separately and instead impose a constant duration across firms.

We first start by estimating the duration of the U.S. corporate sector as a whole. For an infinitely-lived representative firm with constant growth, its duration is simply the inverse of its payout yield.\(^{27}\) This measure averages 3% from 1985 to 2020 (see Equation 24 for the

\[V_t = E_0 \left[ \int_0^\infty e^{-rs}CF_{t+s} \, ds \right] = \frac{CF_t}{r-g}, \]

Differentiating this expression with respect to \(r\), we obtain that the duration of the firm is \(\left| \partial_r \log V_t \right| = 1/(r-g) = V_t/CF_t\). In words, the duration of a firm with constant growth is equal to the inverse of its payout yield.
definition of the payout yield of the corporate sector), which implies an average duration of 35 years. This back of the envelope calculation aligns closely with the findings of van Binsbergen (2020), who estimates a similar duration for the U.S. stock market using dividend strips.

We believe that a duration of 35 years is conservative. Indeed, there are reasons to think that firms owned by successful entrepreneurs have a higher duration than the corporate sector as a whole, since their cash flows are typically negative before turning positive. To account for this fact, we use results from Gormsen and Lazarus (2019), who estimates the duration of public firms using the duration of their realized cash flows. They find that the average duration of the top 20% of the firms in CRSP (sorted according to ex-ante measures of duration) is 46 years. Hence, as a robustness check, we consider an alternative duration calibration of 50 years.

Results. We now use our estimator $\partial r_{\log \theta}$ (defined in Equation 21) to combine our estimates of the average equity payout yield, market leverage, and growth rate of wealth of individuals reaching the top of the wealth distribution. Table 3 contains the result. In our preferred calibration, we obtain a value of $-4.6$, which means that a permanent and unanticipated one percentage point increase in the required return on wealth changes Pareto inequality by $-4.6$ log points. To account for sampling uncertainty, we also provide bootstrapped confidence interval for our sufficient statistic. Note that we can statistically reject the hypothesis that required returns do not matter for Pareto inequality at the 5% level.

We also report our sufficient statistic with two alternative duration calibrations: 20 years and 50 years (relative to our baseline of 35 years). Consistent with the intuition, the sufficient statistic declines monotonically with the duration of the firms founded by entrepreneurs: the higher the duration, the bigger the effect of a change in required returns on valuations, and, as a result, the faster the entrepreneurs’ wealth grows. In Appendix C.1.2, we also examine the sensitivity of our sufficient statistic to potential biases in our measures of the equity payout yield, leverage, and growth rate of wealth.

Table 3: Estimated sufficient statistic $\partial r_{\log \theta}$

<table>
<thead>
<tr>
<th>Duration</th>
<th>Estimate</th>
<th>95% Confidence interval</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>p5</td>
</tr>
<tr>
<td>35 years (baseline)</td>
<td>$-4.6$</td>
<td>$-6.1$</td>
</tr>
<tr>
<td>20 years</td>
<td>$-3.1$</td>
<td>$-4.1$</td>
</tr>
<tr>
<td>50 years</td>
<td>$-6.0$</td>
<td>$-8.1$</td>
</tr>
</tbody>
</table>

Notes. The sufficient statistic is constructed using (21). The 95% confidence interval is constructed as a percentile bootstrap confidence interval using 1000 replications. Data are from Forbes, Compustat, and S-1 filings.

Beyond the top 100. So far, our empirical analysis has focused on the very top of the wealth distribution (i.e., the wealthiest 100 individuals). This is for two reasons. First, it is consistent with the theory: in the case of heterogeneous firm dynamics, our sufficient statistic states that the effect of interest rates on Pareto inequality depends on its effect on the most extreme wealth trajectories (see Equation 15). Second, there is relatively more available data on individuals at the very top, who tend to own well-known (and often public) companies, than for the rest of
the population.

Despite these data limitations, we argue in Appendix C.4 that our cost of capital mechanism should also be relevant well beyond the top 100. First, using data from the Survey of Consumer Finances, we show that roughly half of individuals the top 1% of the U.S. wealth distribution actively manage a firm that they founded. While we have very little information on how much external financing these U.S. entrepreneurs use, we then use cross-country evidence from Kochen (2022) to argue that private firms do rely heavily of both debt (and to a lesser extent) equity financing. Second, and as mentioned in Section 2, our mechanism does not only apply to entrepreneurs who founded firms, but also to any household with a concentrated exposure to a firm that relies on external financing. This includes a large fraction of workers in venture capital backed firms.

4.2 Historical decomposition using the sufficient statistic approach

The required returns on wealth has been declining steadily since the 1980s, a period that saw a rise in wealth inequality. We now use our estimated sufficient statistic to quantify the contribution of declining rates on the rise in Pareto inequality in the U.S. over the 1985–2015 period.

Decline in required returns. In our model, $r$ represents the required return on wealth. In the data, a natural counterpart for $r$ is the required return on business liabilities (i.e., equity and debt claims) issued by the corporate sector. We now propose a simple methodology to estimate this required return using publicly-available macroeconomic data on the nonfinancial corporate sector from the Integrated Macroeconomic Accounts.

We measure the required return of owning business liabilities as its expected return, constructed using two assumptions: (i) revaluation gains (i.e., changes in the corporate sector’s $q$) are zero in expectation and (ii) cash flows are known one period in advance. As detailed in Appendix C.2, these assumptions imply the following formula for the required return on wealth:

$$
\text{Required return} = \frac{\text{Return on capital} - \text{Capital formation rate}}{\text{Tobin’s } q} + \text{Capital formation rate} - \frac{\text{Payout yield}}{\text{Growth rate of cash flows}}.
$$

The required return on wealth can thus be measured as the sum of the payout yield of the corporate sector and its rate of capital formation.\footnote{In particular, Equation 24 holds in the stylized model (see Appendix B.1).} It depends on (i) the net return on capital (i.e., net operating surplus over the replacement cost of capital), (ii) the net rate of capital formation (i.e., net investment over the replacement cost of capital), and (iii) Tobin’s $q$ (i.e., the ratio of the market value of corporations over the replacement cost of capital). In Appendix C.2, we describe the methodology and data in detail. Note that our methodology accounts for inflation, so that everything going forward is in real terms.

One benefit of this methodology is that it allows us to contrast the required return on wealth to the return on capital. In particular, inspecting Equation 24 reveals the fact that the required return on wealth is equal to the return on capital if and only if $q$ is equal to one. Intuitively,
\( q \) represents the ratio between how much investors need to pay in order to acquire a claim on one unit of business capital and its replacement cost. When this ratio is one, the return on wealth for investors coincides with the return on capital.

Figure 3: Returns and valuation for the U.S. nonfinancial corporate sector

Notes. The figure plots Tobin’s \( Q \), the required return on wealth, and the return on capital of the U.S. nonfinancial corporate sector from 1970 to 2020. The construction of each variable is detailed in Appendix C.2. Data are from the Integrated Macroeconomic Accounts.

Figure 3a shows the evolution of the required return on wealth as well as the return on capital. We compare the average in 1980–1985 (pre-transition) to the average in 2015–2020 (end of sample). We find that required returns have declined from 7.8% to 5.1%, implying a \(-2.7\) pp. change in the required return on wealth over the time period. This is quantitatively consistent with the findings in recent papers such as Auclert et al. (2021) and Kuvshinov and Zimmermann (2021), who also use aggregate data to estimate required returns using a similar approach. Our estimate is more conservative (i.e., less negative) than Barkai (2020), who estimates a 5.5 pp. decline, and Mian et al. (2021) who estimate a decline of roughly 3.5 pp.

In contrast, the return on capital exhibits no secular downward trend, and in fact has slightly increased. This consistent with Gomme et al. (2011), who find that the return on capital is mostly flat over time, and Moll et al. (2022), who emphasize the recent rise in the return on capital. The increasing wedge between the return on capital and the required return on wealth has been discussed in Barkai (2020) and Karabarbounis and Neiman (2019): it is directly related to the growth of “pure profits” as a share of GDP (i.e., the part of capital income that cannot be accounted for by the stock of capital and the required return on wealth). Finally, Figure 3b shows the secular rise in Tobin’s \( q \) over the time period, which explains the growing wedge between the return on capital and the required return on wealth (Equation 24).

As discussed in the Stochastic aggregate growth extension in Section 3.2, what matters for Pareto inequality is the decline in required returns relative to the growth rate of the economy (in per-capita terms). In Appendix C.2, we estimate the evolution of required returns deflated using various measures of per-capita growth (i.e., the per-capita capital formation rate or the growth rate of TFP). Accounting for the decline in per-capita growth, we estimate a change in required returns net of per-capita growth of approximately \(-2\) pp.

Note that our required return series accounts for changes in both real interest rates (which matter for debt issuance) and equity valuations (which matters for equity issuance). In Ap-
Pendix C.2.2, we estimate the required return on corporate debt separately and find that it has declined by a similar amount, which justifies our focus on estimating only one required return.

**Rise in Pareto inequality.** Figure 4a plots the evolution of top wealth shares in the U.S. using data from Smith, Zidar and Zwick (2022), who construct wealth estimates based on the capitalization approach. A clear pattern stands out: the top 0.001% wealth share has grown faster than the top 0.01% share, which itself has grown more than the top 0.1% share, and so on. This pattern is a signature of a thickening of the right tail of the wealth distribution (i.e., an increase in Pareto inequality). As discussed in Jones and Kim (2016), if a distribution has a Pareto tail, then Pareto inequality is directly related to the ratio of top shares. Denoting $S(p)$ to be the share of wealth owned by individuals in the top $p \in (0, 1)$, an estimator for Pareto inequality is

$$\hat{\theta}(p) = 1 + \frac{\log(S(p)/S(10p))}{\log 10}. \quad (25)$$

Figure 4b plots the evolution of Pareto inequality using Equation 25 with $p = 0.01\%$ and $p = 0.001\%$, as well as two alternative estimates using Forbes 400 data (i.e., the “mean-min” estimator and the “log rank” estimator, see Appendix C.3 for more details). In each case, we report the log change in Pareto inequality since 1985. The four estimates agree on the broad trend: Pareto inequality has increased substantially since 1985. The fact we obtain a similar trend is reassuring given that the two alternative estimators (i.e., log-rank and mean-min) are purely based on cross-sectional data from the Forbes list, and therefore do not rely on indirect measures of wealth based on the capitalization approach. Taking a simple average of the log change of each estimate implies that Pareto inequality has increased by approximately 22 log points between 1985 and 2015. (See Table 14 in Appendix C.3 for related summary statistics.)

**Sufficient statistic approach.** Given our baseline estimate of a 2 pp. decline in required returns net of per-capita growth, our sufficient statistic implies that the contribution of declining
required returns on Pareto inequality is:

\[
\partial_r \log \theta \times (r_{2015} - r_{1985}) \approx -4.6 \times -2 \text{ pp.} = 9.2 \text{ log points.} \quad (26)
\]

Since the overall change in Pareto inequality was roughly 22 log points over the time period, we conclude that the decline in the required rate of return on wealth accounts for between a third and half of the rise in Pareto inequality. Using the alternative duration assumption of 50 years, as opposed to 35 years, we would obtain that it accounts for more than half of the rise in Pareto inequality.

**Robustness checks.** We conduct two robustness checks. First, we assess the importance of higher-order effects. Note that Equation 26 represents a first-order approximation for the effect of a non-infinitesimal change in the required return \( r \) on Pareto inequality. One may be concerned that this approximation does not capture well the higher-order effects of a 2 pp. change in required returns. To examine this point, we estimate our sufficient statistic using 1985 as a reference year (i.e., focusing on individuals that were at the top of the wealth distribution in 1985). Under certain conditions, the average of the sufficient statistic for the reference years 1985 and 2015 constitutes a second-order approximation for the effect of \( r \) on Pareto inequality. In Appendix C.1.3, we describe this robustness check in details. In short, we do not find much evidence in favor of higher-order effects, which suggests that our first-order approximation is accurate.

Second, we assess the importance of differential declines in the required return on wealth and interest rates on corporate debt. However, as discussed earlier, we find that both returns declined by almost the same amount. Hence, we obtain results that are very similar to (26). See Appendix C.2.2 for the details.

5 Calibrated Model

We now simulate the effect of a decline in the required return \( r \) on top wealth inequality \( \theta \) in a calibrated, general equilibrium model. The goal of this section is fourfold.

First, while our sufficient statistic approach quantifies the effect of an infinitesimal decline in required returns, the model allows us to compute the effect of a non-infinitesimal decline in required returns of 2 pp., as in the data. Second, while our sufficient statistic approach is a comparative static on steady-states of the model, and therefore corresponds to the long-run effect of required returns on Pareto inequality, the calibrated model allows us to characterize the transition dynamics of the wealth distribution. We find that the calibrated model features a relatively high speed of convergence, which is consistent with the data. Third, while our theoretical framework focuses on the change in Pareto inequality, the calibrated model allows us to compute the full change in the wealth distribution. In particular, we report the relative importance of the capital accumulation and the revaluation channels for the share of aggregate wealth owned by different percentiles of the wealth distribution. Fourth, the model allows us to clarify which assumptions are needed to generate a decline in the required return without a corresponding decline in the return on capital, as in the data.
Modelling choices. We study an extended version of the stylized model which combines endogenous investment, adjustable input, leverage, and heterogeneous firm dynamics (see Section 3). In particular, the fact that entrepreneurs produce goods with a combination of capital and labor, the supply of which is fixed in the economy, allows us to incorporate an equilibrium link between the required return \( r \) and the return on capital \( r_{ok} \). We keep the firm side of the model simple, yet rich enough to match the micro evidence from Section 4. In addition to entrepreneurs, we add two other groups of agents: workers and foreigners. The addition of workers is mainly for accounting: it allows us to match the higher duration of entrepreneur wealth relative to aggregate wealth. The addition of foreigners allows us to generate a rise in the demand for domestic assets that originates from abroad.

So far, our analysis has remained agnostic on the causes of declining required returns. The existing literature has emphasized the importance of demand shocks such as increased demand for US assets from abroad (i.e., “global savings glut”) as well as an ageing of the US population and a rise in permanent labor income inequality that increases the domestic demand for assets (i.e., “domestic savings glut”).

For our baseline model experiment, we generate an exogenous asset demand shock driven by foreigners, which pushes down the equilibrium required return. As a robustness check, we also simulate a domestic savings glut, which we operationalize via a decline in the subjective discount factor of domestic agents (i.e., entrepreneurs and workers).

5.1 Environment

As in the stylized model, there is no aggregate risk, agents are infinitively-lived, and population grows at rate \( \eta \). There are two types of domestic agents: a fraction \( \pi \) of newborns are entrepreneurs endowed with a firm while the remaining \( 1 - \pi \) are workers endowed with human capital. In addition, there is a foreign sector that purchases an exogenous amount of domestic financial assets.

Firm problem. Firms are born in a “growth” state and then transitions to a “mature” state at Poisson rate \( \tau \). When the firm transitions to the mature state, its capital jumps by a factor of \( \psi \). The firm problem is to choose the amount of labor \( L \) to hire and the growth rate of capital \( g \). The value functions are the solutions to

\[
\begin{align*}
    r_t V_{0,t}(K) &= \max_{g,L} \left\{ F(K,L) - w_t L - \iota_0(g) K + V_{0,t}'(K) g K + \tau \left( V_{1,t}(\psi K) - V_{0,t}(K) \right) \right\} + V_{0,t}(K), \\
    r_t V_{1,t}(K) &= \max_{g,L} \left\{ F(K,L) - w_t L - \iota_1(g) K + V_{1,t}'(K) g K \right\} + V_{1,t}(K),
\end{align*}
\]

where \( V_{s,t}(K) \) denotes the value of a firm in state \( s \) with capital \( K \), and \( r_t \) is the required return. The subscript \( s = 0 \) corresponds to the growth state while \( s = 1 \) corresponds to the mature state. In terms of the production and investment technology, notice that growth and mature firms only differ in their investment adjustment cost function \( \iota_s(g) \).

---

See Mian et al., 2020 for a recent review of the evidence. In particular, a recurrent finding in the literature is that declining economic growth is not sufficient to explain the decline in \( r \).
We assume the following functional forms:

\[
F(K, L) = K^\alpha L^{1-\alpha}, \quad \iota_s(g) = g + \frac{\chi}{2}(g - g_s)^2.
\]

The production function is a standard Cobb-Douglas function and the adjustment cost function is quadratic. Solving for the optimal investment, we obtain \(g_s = g_s + \frac{1}{\chi}(q_s - 1)\), where \(q_s \equiv V_s(K)/K\) is Tobin’s \(q\). The parameter \(\chi > 0\) thus governs the elasticity of capital with respect to \(q\). We will refer to the limit \(\chi \to \infty\) as the “inelastic capital” case, where the investment rate does not respond to the required return. The state-specific shifters \(g_s\) allow for “Luttmer-rocket” dynamics, where firm growth is initially high and then stabilizes, which allows the model to match the fact that some firms reach the top of the size distribution very fast (Luttmer, 2011).

**Household problem.** At birth, entrepreneurs are endowed with the equity in a new growth firm of size \(\bar{K}\), which is worth \(V_0(\bar{K}) - (1 - \lambda^{-1})\bar{K}\). (We describe the structure of financial markets and the meaning of the book leverage parameter \(\lambda\) shortly.) They have log utility and their subjective discount factor is \(\rho_E\). They are required to maintain all of their wealth invested in the equity of their firm. Their optimal consumption rule is to consume a fixed fraction \(\rho_E\) of their financial wealth.

Workers inelastically supply a unit flow of labor services and earn the equilibrium flow wage \(w\). Their subjective discount factor is \(\rho_L\) and they invest in a diversified portfolio of financial assets. Their optimal consumption rule is to consume a fixed fraction \(\rho_L\) of their total wealth (i.e., financial wealth plus human wealth).

We assume that foreigners invest in a diversified portfolio of domestic financial assets. Let \(S_{F,t}e^{\eta t}\) denote the flow of savings from from abroad at time \(t\), which we treat as exogenous. Our baseline model experiment will consist of perturbing the path of savings by foreigners in order to generate an equilibrium decline in the required return.

**Financial markets.** There are three assets available for trading: a floating rate bond, a levered equity share in the growth firm, and a levered equity share in the mature firm. We assume that firms issue both debt and equity and maintain a fixed book leverage \(\lambda\) (see Section 3). Since there is no aggregate risk, all assets have the same expected return in equilibrium. However, they have different duration profiles (i.e., the bond has zero duration while the equity shares have a positive duration).

Having these three assets allows us to parsimoniously match the higher duration of entrepreneur wealth relative to aggregate wealth, which will discipline the importance of the revaluation channel. Since workers and foreigners are indifferent between investing in any of the assets, we assume that they all hold the same diversified portfolio that comprises all of the assets not held by entrepreneurs.

**5.2 Equilibrium**

We consider a detrended economy where variables are defined in per-capita terms (i.e., multiplied by \(e^{-\eta t}\)). A perfect foresight equilibrium is a sequence of consumption for workers and
entrepreneurs \((C_{L,t}, C_{E,t}) \geq 0\), a growth rate of capital for both types of firms \((g_{0,t}, g_{1,t}) \geq 0\), a labor demand for both types of firms \((L_{0,t}, L_{1,t}) \geq 0\), and a level of capital for both types of firms \((K_{0,t}, K_{1,t}) \geq 0\), such that (i) consumption, labor demand, and capital growth solves the worker, entrepreneur, and firm problems and (ii) the labor and product markets clear:

\[
\sum_s L_{s,t} = 1 - \pi, \tag{29}
\]

\[
\sum_s F(K_{s,t} L_{s,t}) = C_{L,t} + C_{E,t} - S_{F,t} + \sum_s K_{s,t} (g_{s,t}). \tag{30}
\]

In Appendix D.1, we provide an analytical characterization of the equilibrium.

**Neoclassical growth model as a special case.** In Appendix D.2 we show that the model nests the neoclassical growth model for a particular configuration of the model parameters, where capital is fully elastic \((\chi = 0)\), there is no firm heterogeneity \((\psi = 0)\), all agents are workers \((\pi = 1)\), and there is no population renewal \((\eta = 0)\). However, in order to match the empirical evidence from Section 5.3 (i.e., the micro moments related to wealth dynamics as well as the aggregate wedge between the return on capital and the required return on wealth), our calibration strategy selects a set of parameters that differs starkly from the “neoclassical growth calibration”.

**5.3 Calibration**

We calibrate the model by targeting moments for the US economy over the 1985–2015 period. We target a 7% required at the initial steady-state (i.e., roughly the expected return in 1985 net of trend growth, see Appendix Table 12). Our model experiment will be to feed an exogenous rise in foreign savings that implies an equilibrium decline in the required return to 5%. Depending on the moment we want to match, we use the steady-state of the model associated with a required return of \(r = 7\%\) (i.e., 1985 moments) or the steady-state of the model associated with a required return of \(r = 6\%\) (i.e., 1985–2015 average moments).

First, we set the values for \((\alpha, \pi, \chi)\) externally. The capital share is set to its standard value of \(\alpha = 1/3\) and the share of entrepreneurs in the economy is set to \(\pi = 0.15\), which is roughly equal to the share of business owners in the US (see Table 1 of Cagetti and De Nardi, 2006). We set the capital adjustment cost parameter to \(\chi = +\infty\) to make aggregate capital inelastic. We do so in order to match the fact that investment and the return on capital does not appear to have declined in the US, despite a large rise in Tobin’s q. This will serve as our baseline, but we also consider alternative calibrations with finite values for \(\chi\) in Section 5.6.

Second, we use the the remaining 9 parameters \((g_{0,t}, g_{1,t}, \tau, \psi, \lambda, \bar{K}, \eta, \rho_{L}, \rho_{E})\) to match (i) the four conditional micro moments that enter the sufficient statistic (i.e., equity payout yield, growth rate of wealth, duration, and market leverage) and (ii) five macro moments (Pareto inequality, net return on capital, depreciation rate, aggregate duration, and the net foreign asset position). The conditional micro moments are taken directly from Table 2. Those are important moments to match: they fully determine the long-run response of Pareto inequality to the required return. While these moments are measured for individuals at the top of the wealth
Table 4: Targeted moments

<table>
<thead>
<tr>
<th>Moment</th>
<th>Period</th>
<th>Model</th>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conditional micro moments</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Equity payout yield</td>
<td>1985–2015</td>
<td>−0.024</td>
<td>−0.024</td>
</tr>
<tr>
<td>Growth rate of wealth</td>
<td>1985–2015</td>
<td>0.31</td>
<td>0.31</td>
</tr>
<tr>
<td>Market leverage</td>
<td>1985–2015</td>
<td>1.37</td>
<td>1.37</td>
</tr>
<tr>
<td>Duration</td>
<td>1985–2015</td>
<td>34</td>
<td>35</td>
</tr>
<tr>
<td>Macro moments</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Return on capital</td>
<td>1985</td>
<td>0.071</td>
<td>0.070</td>
</tr>
<tr>
<td>Depreciation rate</td>
<td>1985–2015</td>
<td>0.081</td>
<td>0.080</td>
</tr>
<tr>
<td>Pareto inequality</td>
<td>1985–2015</td>
<td>0.60</td>
<td>0.60</td>
</tr>
<tr>
<td>Aggregate duration</td>
<td>1985–2015</td>
<td>21</td>
<td>20</td>
</tr>
<tr>
<td>NFA to domestic wealth</td>
<td>1985–2015</td>
<td>−0.05</td>
<td>−0.05</td>
</tr>
</tbody>
</table>

\[(g_0, g_1, \tau, \psi, \lambda, \bar{K}, \eta, \rho_L, \rho_E) = (0.370, −0.024, 0.409, 0.503, 2.394, 9.878, 0.110, 0.043, 0.035)\].

distribution in 2015, they are backward-looking lifetime averages, so we use the model’s \(r = 6\%\) steady-state as the theoretical counterpart (1985–2015 moments).

For the macro moments, we first target a net return on capital of 0.07 in the \(r = 7\%\) steady-state of the model (1985 moment), so that the model initially matches the fact that the expected return on wealth is equal to the return on capital at the beginning of the sample (see Appendix Table 12). All of the other macro moments are associated with the \(r = 6\%\) steady-state of the model (1985–2015 moments). We target a depreciation rate of 8% and a level of Pareto inequality of 0.6 (see Appendix Table 12). For the duration of aggregate wealth, we rely on evidence from Greenwald et al. (2021), who estimate a duration of roughly 20 years.\(^{30}\) Finally, we target a net foreign asset position to domestic wealth of \(-5\%\).\(^{31}\) Table 4 reports the targeted moments in the data and in the model.

5.4 Model experiment

Our model experiment consist of feeding, a every time \(t \geq 0\), a sequence of MIT shocks to the path of foreign savings \((dS_F)^s\) in order to generate a smooth decline in the required return from an initial steady-state value of \(r = 7\%\) to a long-run value of \(r = 5\%\).\(^{32}\) We construct the path of foreign saving shocks so that, for every \(t \geq 0\), the instantaneous change in the required return \(dr_t\) is both fully unexpected and believed to be permanent.\(^{33}\) This is roughly consistent with the empirical evidence. For instance, Farmer et al. (2021) shows that professional

\(^{30}\)Unlike in our model, household wealth in the U.S. is not only composed of corporate liabilities (i.e., corporate equities and debts), but also of real estate and government liabilities. In the context of our model experiment, however, what is important is to match the duration of the wealth held at the top of the wealth distribution and the duration of aggregate wealth, as the difference between the two determines the size of the revaluation channel.

\(^{31}\)In the model, domestic households do not own foreign assets. Hence, the net foreign asset position (in absolute term) is the value of domestic assets held by foreigners. Using data from the Integrated Macroeconomic Accounts, we find that this ratio went from nearly zero in 1985 to \(-10\%\) in 2015, with a midpoint of \(-5\%\) over the sample. See Appendix C for details.

\(^{32}\)In practice, we target a path for the required return given by \(r_t = 0.07e^{-\phi t} + 0.05(1 - e^{-\phi t})\), with \(\phi = 7.5\%\).

\(^{33}\)To be precise, at every time \(t \geq 0\), the contemporaneous consumption, investment, and hiring decisions \((C_L, C_E, S_H, S_I, L_0, L_1)\) are part of a perfect foresight equilibrium with constant required return going forward: \(r_{t+s} = r_t\) for all \(s \geq 0\).
forecasters have been consistently forecasting a flat path for short-term interest rates over the 1985–2015 period, despite the fact that short-term rates were a continuous downward path.

The model experiment thus consists of simulating the response of our model economy to a global savings glut (i.e., a rise in savings that originates outside of the domestic economy). We study a time period of 40 years following the beginning of the shock sequence. In Appendix D.1, we provide a detailed description of the equilibrium construction as well as the numerical algorithms used to solve for the equilibrium path.

Figure 5: Returns and valuations (model experiment)

The left panel of Figure 5 shows the evolution of the required return over time. The sequence of shocks starts at \( t = 0 \), after which the required return on wealth declines monotonically. In contrast, the return on capital remains constant as capital is fully inelastic in this calibration (this assumption will be relaxed in Section 5.5). By construction, the paths of the required return on wealth and the return on capital over the period \( t \in [0, 30] \) follow closely their empirical counterparts over the 1985–2015 period (see Figure 3).

Figure 5 also plots the evolution of the realized return on aggregate wealth. Note that it initially increases. To understand why, it is useful to express the instantaneous realized return as the sum of the required return and the unexpected revaluation of assets. In the model experiment, we have that:

\[
\frac{dR_t}{R_t} = \underbrace{r_t \ dt}_{\text{Realized return}} + \underbrace{dq_t}{q_t} - \underbrace{E_t \left[ \frac{dq_t}{q_t} \right]}_{\text{Revaluation}}. 
\]

Since Tobin’s q increases over the transition (see Figure 5b), the revaluation term is positive. Note that the magnitude of this term is empirically disciplined by the fact our model targets the duration of aggregate wealth in the data.\(^{34}\)

Figure 6a shows the evolution of top wealth shares, all normalized to one at \( t = 0 \). The top shares are calculated numerically and expressed as shares of aggregate wealth. For simplicity, we assumed that all workers are identical, and, as a result, only entrepreneurs are in the top

\(^{34}\)Our calibration matches a duration of aggregate wealth of 20. During a short time period \( dt \), the innovation in the required return \( dr_t \) is believed to be permanent. As a result, in a calibration with inelastic capital, the
Notice that the top 0.1% increases more than the top 1%, and the top 0.01% increases more than the top 0.1%, and so on, which we precisely what we observe in the data (see Figure 4). To better visualize this effect, Figure 6b shows the evolution of Pareto inequality (calculated using the top share estimator with \( p = 0.1\% \), see Equation 25), expressed in log and normalized to zero at \( t = 0 \). Pareto inequality increases steadily and roughly converges to its long-run value after 40 years. Compared to the data, the model roughly matches the rise in the top 1% wealth share, but undershoots the rise in the top 0.001% (see Figure 4a), which is consistent with the fact that our proposed mechanism explains roughly half of the rise in Pareto inequality (see Section 4).

We also plot a dashed grey line showing the long-run level of Pareto inequality predicted by the sufficient statistic approach. This line coincides almost exactly with the long-run limit of Pareto inequality in the model. This suggests that, at least in this model, our sufficient statistic constitutes a very good approximation of a non-infinitesimal change in the required return on wealth on Pareto inequality. More precisely, while the sufficient statistic approach holds exactly in the model for small changes in \( r \) (i.e., it relies on a first-order approximation), we obtain a slightly higher response in the model in the long-run due to higher-order effects.

Finally, the calibrated model generates a relatively fast convergence of Pareto inequality to its long-run steady-state. As discussed in Gabaix et al. (2016), this comes from the presence of high-growth types in our model: indeed, in our model, some agents (i.e., entrepreneurs owning growth firms) reach the right tail of the wealth distribution quickly, as their wealth grows at an annual rate of 31% (see Table 4). Still, note that our estimate of Pareto inequality revaluation term is given by

\[
\frac{dq_t}{q_t} - \mathbb{E}_t \left[ \frac{dq_t}{q_t} \right] \approx -20 \times dr_t.
\]

See Footnote 6 for a definition of duration.

\(^{35}\)Given that a fraction \( \pi = 15\% \) of the population are workers, our top 1% group corresponds to the top 1/15 \( \approx 6.7\% \) of entrepreneurs.

\(^{36}\)Formally, the sufficient statistic changes over the time period, as the lifetime average equity payout yield, leverage, duration, and growth rate of entrepreneurs in the top changes in response to the decline in the required return. See Appendix C.1.3 for a discussion of higher-order effects.
(measured as the ratio between the wealth share of the top 0.01% relative to the top 0.1%) remains constant in the first ten years. This reflects the fact that, initially, all percentiles benefit similarly from a decline in the required return on wealth. It is only when the new generation of entrepreneurs, born in the new low interest rate environment, reach the top 0.1% that Pareto inequality start increasing.

**Capital accumulation versus revaluation channel.** As discussed in Section 3.3, a lower required return tends to increase the share of aggregate wealth owned in a top percentile through two distinct channels. First, it increases the relative quantity of capital owned by entrepreneurs who raise external financing, as they now face a lower cost of capital (a “capital accumulation” channel). Second, it can increase the valuation of the capital owned by entrepreneurs relative to the valuation of the aggregate capital (a “revaluation” channel).

We now assess the relative contribution of each channel in our model economy by decomposing the cumulative growth of the share of wealth owned by a top percentile. Formally, the average wealth owned by a top percentile $p$ at time $t$ can be written $q_{\lambda p t} E_{p t}$ where $E_{p t}$ is the average quantity of book equity owned by the top percentile and $q_{\lambda p t}$ is the weighted-average valuation of this equity. The average wealth in the economy, can be written $q_t K$ where $K$ is aggregate capital per capita and $q_t$ is the weighted-average valuation of capital in the economy. Given these notations, we can decompose the cumulative growth of the share of aggregate wealth owned by a top percentile $p$ between 0 and $t$ into a “capital accumulation” and a “revaluation” channel:

$$
\log \left( \frac{q_{\lambda p t} E_{p t}/(q_t K)}{q_{\lambda p 0} E_{p 0}/(q_0 K)} \right) = \log \left( \frac{E_{p t}}{E_{p 0}} \right) + \log \left( \frac{q_{\lambda p t}}{q_{\lambda p 0}} \right) - \log \left( \frac{q_t}{q_0} \right).
$$

Figure 7: Disentangling capital accumulation and revaluation (model experiment)

Figure 7a plots the cumulative growth of the top 0.1% in the model experiment, as well as the cumulative contribution of the capital accumulation and of the revaluation channels. We find that the revaluation channel is positive. The reason is twofold: (i) entrepreneurs at the top
of the wealth distribution hold levered positions in firms, and (ii) they tend to hold firms with a higher duration than the average firm in the economy. Quantitatively, in the $r = 6\%$ steady-state of the model, the levered duration (i.e., duration times market leverage) of the assets held by the top 0.1% of agents $|\partial_r \log q_{\lambda pt}|$ is 33 while the duration of aggregate wealth $|\partial_r \log q_t|$ is 21. Hence, a first-order approximation for the long-run contribution of the revaluation channel is $(|\partial_r \log q_{\lambda pt}| - |\partial_r \log q_t|) \times \Delta r = (33 - 21) \times 2 \text{ pp.} = 24 \text{ log points}$, which is precisely what we obtain in Figure 7a. Note that this number is tightly disciplined by our calibration strategy, where we match the duration of aggregate wealth as well as the duration and market leverage of individuals reaching the top of the wealth distribution (see Table 4).

We also find that the contribution of the revaluation channel is builds up more quickly, relative to the capital accumulation channel. The reason is that revaluation gains are immediate: when the required return declines, the market value of capital jumps up, which affects the distribution of wealth on impact. In contrast, the capital accumulation channel builds up slowly over time, as it takes time for existing entrepreneurs to raise new outside financing via equity and debt issuance.

Figure 7b plots the long-run contribution of each channel for the top 1%, 0.1%, 0.01%, and 0.001% wealth shares (i.e., from the initial steady-state to the terminal steady-state). The contribution of the revaluation channel is approximately the same for all top percentiles. The reason is that the composition of individuals in each of these top groups is approximately the same (i.e., the proportion of entrepreneurs owning growth firms versus mature firms), and, therefore, they experience similar revaluation gains. In contrast, the contribution of the capital accumulation channel increases with the top percentile (i.e., it is higher for the top 0.001% than for the top 0.01%, and so on). As discussed in Section 3.3, this reflects the fact that wealthier individuals spend more time in the high growth state over their lifetime, and, as a result, they benefit more from a lower cost of capital as they go through a higher number of funding rounds.

To sum up, we find that both the revaluation and the capital accumulation channels contribute to the rise in top wealth shares. One key difference, however, is that the relative importance of the capital accumulation channel increases the thickness of the right tail of the wealth distribution. As discussed formally in Section 3.3, this is because the rate of capital accumulation is the only thing that matters for Pareto inequality.

The fact that the revaluation channel only explains part of the rise in top wealth shares in our model is consistent with Saez and Zucman (2016) and Mian et al. (2020), who argue that the mechanical effect of the revaluation of assets owned by the rich cannot fully explain the rise in top wealth shares over the past four decades. They impute this difference to a rise in the “synthetic” saving rate of individuals in top percentiles. In our model, however, the consumption rate of entrepreneurs is fixed. Instead, the difference is driven by an increase in the flow of entrepreneurs reaching the top of the wealth distribution. We examine this distinction in more details in Appendix D.5.

$^{37}$The fact that the relative proportion of each type converges in the right tail is a general result in random growth models (see, for instance, Proposition 4.1 in Gouin-Bonenfant and Toda, 2022).
5.5 Domestic savings glut

In the baseline model experiment, we generate an equilibrium decline in $r$ by feeding in an exogenous rise in savings by foreigners. We now consider an asset-demand shock that originates domestically (i.e., a domestic savings glut). We do so by changing the subjective discount factor $\rho$ of domestic agents. This captures, in a reduced-form way, a number of forces, such as rising longevity, that pushes up the desire to save. The key difference with the baseline model experiment is that a decline in $\rho$ has a direct on top wealth inequality, a force that we now quantify.

To implement the model experiment, we consider a model extension where workers and entrepreneurs have time-varying subjective discount factors and where the flow of savings by foreigners is constant over time. We use the same calibration as in the baseline model experiment, but shift the subjective discount factors of domestic workers and entrepreneurs ($\rho_L, \rho_E$) to generate a 2 pp. equilibrium decline in the required return from 7% to 5%. In Appendix D.4, we describe the model experiment and solution method in details.

Table 5 reports the long-run change in the required return, the subjective discount factor, as well as the change in (log) Pareto inequality. Overall, we find that the rise in Pareto inequality is roughly 1.5 times larger than in the baseline model. To understand the forces at play, it is instructive to use an comparative statics formula for the change in Pareto inequality in response to an infinitesimal change in the subjective discount factor $d\rho$ and the required return $dr$:

$$d \log \theta = \frac{1 + \text{Market leverage} \times (\text{Equity payout yield} \times \text{Duration} - 1)}{\text{Growth rate of wealth}} dr + \frac{-1}{\text{Growth rate of wealth}} d\rho.$$  (33)

The formula expresses the change in Pareto inequality as a linear function of the change in the required return $r$ and the change in the subjective discount factor $\rho$. See Appendix D.4 for the derivation. In the model experiment, the required return $r$ declines by 2 pp. while the subjective discount factors decline by roughly 1.4 pp. However, the sensitivity of Pareto inequality to required returns is higher than its sensitivity to the subjective discount factor. The reason is that a change in $\rho$ moves the growth rate of wealth of wealth one-for-one for all entrepreneurs, while a change in $r$ affects the growth rate of successful entrepreneurs more than one-for-one, due to the fact that these entrepreneurs use a lot of external financing and own high-duration firms (i.e., $1 + \text{Market leverage} \times (\text{Equity payout yield} \times \text{Duration} - 1) = 1.52$, see Table 4).

The reason why a 1.4 pp. decline in the subjective discount factors of both workers and entrepreneurs leads to a 2 pp. decline in the required return is that there is a reallocation of wealth towards entrepreneurs who have a lower subjective discount factor that workers (see Table 4).
5.6 The elasticity of capital

In general, the return on capital is an endogenous variable that depends on aggregate capital, labor, and investment. In the baseline calibration of the model, we hardwired a constant return on capital by making capital completely inelastic (i.e., we set the investment adjustment cost to $\chi = +\infty$). We now consider three alternative calibrations where we use the parameter $\chi$ to match three targets for the decline of the return on capital (i.e., 0.5, 1, and 1.5 pp.). We refer to these calibrations as “low-elasticity”, “medium-elasticity”, and “high-elasticity” calibrations. These targets provide a range of intermediate value between the baseline model (i.e., no decline in the return on capital) and the neoclassical growth model (i.e., 2 pp. decline in the return on capital). The calibration strategy remains otherwise identical, but we have one additional model parameter (i.e., $\chi$) and one additional moment (i.e., the decline in the aggregate return on capital). See Appendix D.6 for the calibration table.

In calibrations with elastic capital, there are two forces that generate a decline in the return on capital. Applying our definition of the return on capital (17) for a firm in state $s$, and substituting the expression for optimal labor, we have that

$$\text{rok}_s \equiv \alpha \left( \frac{K_s}{L_s} \right)^{\alpha - 1} - \frac{\chi}{2} \left( g_s - \bar{g}_s \right)^2, \quad (34)$$

The first term is production efficiency, which in this model is the marginal product of capital, and the second term is investment efficiency. The direct effect of a decline in the required return $r$ is that it increases Tobin’s $q$ and leads to a rise in investment (recall that the policy function is $g_s = \bar{g}_s + \frac{1}{\chi} (q_s - 1)$). As result, the aggregate capital stock increases. But since labor supply is fixed at the aggregate level, this implies a rise in the capital to labor ratio, which puts downward pressure on the return on capital.$^{39}$ The second force that depresses the return on capital is that investment efficiency declines. Indeed, as firm-level investment $g$ increases, adjustment costs become more severe due to the convexity of $\iota_s(g)$, which decreases the amount of net capital income per unit of capital.

Table 6: Model experiment with elastic capital (long-run, percentage points)

<table>
<thead>
<tr>
<th>Model</th>
<th>$\Delta r$</th>
<th>$\Delta \text{rok}$</th>
<th>$\Delta \log \theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baseline</td>
<td>-2.0</td>
<td>0</td>
<td>10.8</td>
</tr>
<tr>
<td>Low-elasticity</td>
<td>-2.0</td>
<td>-0.5</td>
<td>9.5</td>
</tr>
<tr>
<td>Medium-elasticity</td>
<td>-2.0</td>
<td>-1.0</td>
<td>8.1</td>
</tr>
<tr>
<td>High-elasticity</td>
<td>-2.0</td>
<td>-1.5</td>
<td>7.0</td>
</tr>
<tr>
<td>Neoclassical growth model</td>
<td>-2.0</td>
<td>-2.0</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 6 reports the long-run response of the aggregate return on capital ($\Delta \text{rok}$) as well as the long-run increase in (log) Pareto inequality (i.e., $\Delta \log \theta$) in each model calibration. The key takeaway is that, the more elastic capital is, the lower the rise in Pareto inequality. For instance, in the baseline model, we have that Pareto inequality increases by 10.8 log points. In contrast, in the high-elasticity calibration, the corresponding value is 7 log points, which is roughly one-third of the baseline value.

$^{39}$Since both types of firms have the same production function, they both employ as much labor per unit of capital. As a result, they both have the same production efficiency.
third lower. The reason is that, for a given cost of capital \( r \), a lower return on capital reduces the growth rate of wealth for successful entrepreneurs (i.e., those who own growth firms for a long period of time). Indeed, Equation 19 in the stylized model makes it clear that both the return on capital and the cost of capital matter for the realized return on wealth of successful entrepreneurs.

Overall, we conclude that, for reasonable model calibrations (i.e., calibrations that imply a moderate decline in the return on capital), a decline in \( r \) increases Pareto inequality materially. In Appendix D.6, we compare the implied values of \( \chi \) in the elastic capital calibrations to existing evidence from investment regressions.

6 Conclusion

This paper studies the long-run relationship between Pareto inequality in wealth (i.e., inequality between wealthy individuals) and the required rate of return on wealth. We make three distinct contributions. First, we show theoretically that low rates increase top wealth inequality as long as individuals reaching the top of the wealth distribution are “net borrowers” rather than “net lenders”. Second, we derive a sufficient statistic for the effect of lower rates on top wealth inequality (as measured by the Pareto exponent of the wealth distribution). It depends on three key moments: the average growth rate of wealth for individuals reaching the top of the wealth distribution as well as the average payout yield and leverage of the firms that they own. Third, we collect new data on the wealth trajectory of the top 100 wealthiest individuals in the U.S., which we use to estimate our sufficient statistics.

Overall, our results indicate that the direct effect of lower rates on top wealth inequality is large: the 2% decline in required returns that we estimate from 1985 to 2015 accounts for between a third and half of the rise in top wealth inequality. This finding is guided by the observation that, in the U.S., entrepreneurs reaching the top of the wealth distribution rely heavily on external financing. Technology and institutions presumably affect the extent to which successful firms rely on external financing. In particular, the effect of interest rates on top wealth inequality may be drastically different across countries and time periods. We view our sufficient statistic approach as a first step in understanding this heterogeneous effect.

Taking a step back, one important message of our paper is that the right tail of the U.S. wealth distribution is determined by the wealth dynamics of entrepreneurs, rather than by the wealth dynamics of rentiers. We develop a simple model as well as analytical methods that allow us to quantify the effect of changes in entrepreneur wealth dynamics on Pareto inequality. We believe that this set of tools could prove useful to shed light on other factors that might be driving the recent rise in top wealth inequality such as changes in the nature of technology (e.g., Kaplan and Rauh, 2010; Jones and Kim, 2016; Moll et al., 2022) and changes in corporate taxation (e.g., Kaymak and Poschke, 2016; Hubmer et al., 2020).

Finally, the idea that low interest rates increase top wealth inequality complements a growing literature which argues that high inequality puts downward pressure on required rates of return in equilibrium (see, for instance, Straub, 2019 for the interest rate or Gollier, 2001; Toda and Walsh, 2019; Gomez, 2016 for the equity risk premium). Taken together, this suggests that high wealth inequality and low required rates of return can be mutually reinforcing in the
long-run, an idea we leave for future research.

References


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Online Appendix (not for publication)

A Appendix for Section 2

A.1 Proofs

We first prove the following lemma.

**Lemma 5.** For \( r \in (g - \tau, \rho + \tau) \), denote

\[
\theta_E(r) = \frac{g - \frac{1}{q} - \rho}{\tau + \eta}, \quad \theta_R(r) = \frac{r - \rho}{\eta}, \quad \theta(r) = \max(\theta_E, \theta_R).
\]

There exists \( r^* \in (g - \tau, \rho + \tau) \) such that \( \theta_E(r^*) = \theta_R(r^*) > 0 \). As a result, \( \theta(\cdot) \) is decreasing in \( r \) over \( r \in (g - \tau, r^*) \) and increasing in \( r \) over \( r \in (r^*, \rho + \tau) \).

**Proof of Lemma 5.** We now compute the value of \( r^* \).

\[
\begin{align*}
\theta_E(r^*) &= \theta_R(r^*) \\
\iff r^* + \tau \left(1 - \frac{1}{q(r^*)}\right) - \rho &= \frac{r^* - \rho}{\eta} \\
\iff 1 - \frac{1}{q(r^*)} &= \frac{r^* - \rho}{\eta} \\
\iff 1 - \frac{r^* + \tau - g}{\tau - i} &= \frac{r^* - \rho}{\eta} \\
\iff (\tau - i - r^* + g)\eta &= (r^* - \rho)(\tau - i) \\
\iff r^* &= \frac{\eta}{\eta + \tau - i}(g - \tau) + \frac{\tau - i}{\eta + \tau - i}(\rho + \eta)
\end{align*}
\]

Note that the denominator is positive as \( \tau > i \). This equation says that \( r^* \) can be seen as a weighted average between the lower bound \( g - \tau \) and the upper bound \( \rho + \eta \), and, in particular, \( r \in (g - \tau, \rho + \tau) \). At this value \( r^* \), we have

\[
\theta_E(r^*) = \theta_R(r^*) = \frac{g - i - \rho}{\eta + \tau - i}
\]

The assumption that \( \rho < g - i \) ensures that the numerator is positive and the assumption that \( i < \tau \) ensures that the denominator is positive for all \( \eta > 0 \). We therefore have that \( \theta_R(r^*) = \theta_E(r^*) > 0 \). This implies that \( \theta(r) \) is positive on \( (g - \tau, \rho + \eta) \).

\[
\Box
\]

**Proof of Proposition 2.** Let \( p_R(W), p_E(W) \) denote, respectively, the stationary densities of wealth for renters and entrepreneurs normalized so that \( \int (p_E(W) + p_R(W)) \, dW = 1 \). Denoting by \( \delta(\cdot) \) the Dirac function, the Kolmogorov Forward Equations implies the following system of ODES

\[
\begin{align*}
0 &= -\partial_W \left( (g - \frac{i}{q} - \rho) W p_E(W) \right) + \eta \delta(W - q) - (\tau + \eta) p_E(W), \\
0 &= -\partial_W \left( (r - \rho) W p_R(W) \right) + \tau p_E(qW) - \eta p_R(W).
\end{align*}
\]

Consider \( \theta_E, \theta_R, \) and \( r^* \) defined in Lemma 5. Denote \( r_R < r^* \) such that \( \theta_R(r_R) = 0 \) and \( r_E > r^* \) such
that \( \theta_E(r_E) = 0 \). We now solve for the densities \( p_E(W) \) and \( p_R(W) \):

\[
p_E(W) = \frac{\eta}{\tau + \eta} \times \begin{cases} 
\frac{1}{\theta_E - \theta_R} \left( W^{-\frac{1}{\theta_E}} \cdot W^{-\frac{1}{\theta_R}} \right) & \text{if } r \in (g - \tau, r_E) \quad \text{(i.e., } \theta_E > 0) \\
\delta(W - q) & \text{if } r = r_E \quad \text{(i.e., } \theta_E = 0) , \\
\frac{1}{\theta_E - \theta_R} \left( W^{-\frac{1}{\theta_E}} \cdot W^{-\frac{1}{\theta_R}} \right) & \text{if } r \in (r_E, \rho + \tau) \quad \text{(i.e., } \theta_E < 0) 
\end{cases}
\]

\[
p_R(W) = \frac{\tau}{\tau + \eta} \times \begin{cases} 
\frac{1}{\theta_E - \theta_R} \left( W^{-\frac{1}{\theta_E}} \cdot W^{-\frac{1}{\theta_R}} \right) & \text{if } r \in (g - \tau, r_R) \quad \text{(i.e., } \theta_R < 0 < \theta_E) \\
\frac{1}{\theta_R} W^{-\frac{1}{\theta_R}} - 1_{W \leq 1} & \text{if } r = r_R \quad \text{(i.e., } 0 = \theta_R < \theta_E) \\
\frac{1}{\theta_R} W^{-\frac{1}{\theta_R}} - 1_{W \geq 1} & \text{if } r \in (r_R, r^*) \quad \text{(i.e., } 0 < \theta_R < \theta_E) \\
\frac{1}{\theta_R} W^{-\frac{1}{\theta_R}} - 1_{W \geq 1} & \text{if } r = r^* \quad \text{(i.e., } 0 < \theta_R = \theta_E) \\
\frac{1}{\theta_R - \theta_E} \left( W^{-\frac{1}{\theta_R}} \cdot W^{-\frac{1}{\theta_E}} \right) & \text{if } r \in (r^*, r_E) \quad \text{(i.e., } 0 < \theta_E < \theta_R) \\
\frac{1}{\theta_R - \theta_E} \left( W^{-\frac{1}{\theta_R}} \cdot W^{-\frac{1}{\theta_E}} \right) & \text{if } r = r_E \quad \text{(i.e., } 0 = \theta_E < \theta_R) \\
\frac{1}{\theta_R - \theta_E} \left( W^{-\frac{1}{\theta_R}} \cdot W^{-\frac{1}{\theta_E}} \right) & \text{if } r \in (r_E, \rho + \eta) \quad \text{(i.e., } \theta_E < 0 < \theta_R) 
\end{cases}
\]

Note that our assumptions are general enough that they do not necessarily imply that \( r_R > g - \tau \) or \( r_E < \rho + \eta \). Accordingly, the notation \((a, b)\) should be understood as the empty set when \( a \geq b \). Now, we have \( \mathbb{P}(\text{Wealth} > W) = \int_{W}^{\infty} (p_E(W') + p_R(W')) dW' \). Combining the closed-form expressions for \( p_E(\cdot) \) and \( p_R(\cdot) \) gives us the following asymptotic equivalence result:

\[
\mathbb{P}(\text{Wealth} > W) \sim \begin{cases} 
\left( \frac{\eta}{\tau + \eta} + \frac{\theta_R}{\theta_E - \theta_R} \right) W^{-\frac{1}{\theta_R}} & \text{if } r \in (g - \tau, r^*) \quad \text{(i.e., } \theta_R < \theta_E) \\
\frac{\tau}{\tau + \eta} \log(W) W^{-\frac{1}{\theta_R}} & \text{if } r = r^* \quad \text{(i.e., } \theta_R = \theta_E) \\
\frac{\tau}{\tau + \eta} \frac{\theta_R}{\theta_E - \theta_R} W^{-\frac{1}{\theta_R}} & \text{if } r \in (r^*, r_E) \quad \text{(i.e., } \theta_R > \theta_E) 
\end{cases}
\]

as \( W \to \infty \). By the definition of a Pareto exponent (Definition 1), we obtain that Pareto inequality is given by \( \theta = \max(\theta_E, \theta_R) \).

Finally, denote by \( \pi_E(W) = p_E(W) / (p_E(W) + p_R(W)) \) the relative mass of entrepreneurs a wealth level \( W \). The closed-form expressions for the densities \( p_E(\cdot) \) and \( p_R(\cdot) \) obtained above imply that

\[
\lim_{W \to +\infty} \pi_E(W) = \begin{cases} 
\frac{1}{1 + \frac{1}{\theta_R} \cdot \frac{\theta_E}{\theta_E - \theta_R}} & \text{if } r \leq r^* \\
0 & \text{if } r > r^* 
\end{cases}
\]

\( \square \)

### A.2 Relationship to Wold-Whittle

We now describe how our modeling choices differ from the canonical model of wealth inequality. Then, we explain why our model nests Wold and Whittle (1957) when \( \tau \to \infty \) (all agents are rentiers).

Wold and Whittle (1957) provides the first theoretical treatment relating the Pareto exponent of the wealth distribution to the return on wealth. They study a model of wealth accumulation where households are born with an initial level of wealth, which then grows at some positive growth rate until the household dies (at some common Poisson rate). In a steady-state, the level of Pareto inequality in their model is given by the ratio of the growth rate of wealth to the mortality rate. Assuming, as we do in the stylized model, that individuals consume a fixed fraction \( \rho \) of their wealth, and denoting the
mortality rate by $\eta$. Pareto inequality in Wold-Whittle is $\theta = \frac{r - \rho}{\eta}$, as long as $r > \rho$ (otherwise the wealth distribution would not have a right tail). In contrast, the formula that we obtain in our model is

$$\theta = \max \left( \frac{g - \frac{1}{\rho} - \rho}{\eta + \tau}, \frac{r - \rho}{\eta} \right).$$

However, taking the Poisson rate $\tau$ at which entrepreneurs become rentiers to infinity gives

$$\lim_{\tau \to \infty} \theta = \frac{r - \rho}{\eta}. \tag{37}$$

Hence, our model obtains the Wold-Wittle world as a limiting case.

Note that, in Wold-Wittle, inequality in wealth is entirely the result of inequality in age. In contrast, our model also features return heterogeneity. While entrepreneurs and rentiers earn the same return in expectation, entrepreneurs whose tree keeps on growing earn a different return than rentiers, to compensate for the risk of the tree blossoming. The main point of our paper is that these heterogeneous returns have a different exposure to the interest rate $r$: successful entrepreneurs benefit from a lower $r$ at the expense of rentiers.

A.3 Closing the economy

Agents. Suppose that the economy now also includes “workers”. Workers have log utility with a subjective discount factor $\rho_L$. Like entrepreneurs, workers are also born with trees, but they immediately sell them and instead hold a diversified portfolio of trees. Denote by $\pi$ the proportion of newborn that are entrepreneurs.

State variables. Denote by $K$ the average size of a tree in the economy. The law of motion for $K$ is

$$\dot{K} = (g - \tau)K + \eta(1 - K). \tag{35}$$

Denote $W_{ER}$ to be the per-capita wealth of entrepreneurs and rentiers, $W_L$ to be the per-capita wealth of workers, and $x = \pi W_{ER}/(\pi W_{ER} + (1 - \pi)W_L)$ to be the fraction of aggregate wealth owned by entrepreneurs and rentiers (as opposed to workers). The law of motion of $x$ is given by:

$$\dot{x} = x(1 - x) \left( \frac{W_{ER}}{W_{ER}} - \frac{W_L}{W_L} \right),$$

$$= x(1 - x) \left( (r - \rho + \eta \left( \frac{\pi}{xK} - 1 \right) ) - (r - \rho + \eta \left( \frac{1 - \pi}{(1 - x)K} - 1 \right) ) \right),$$

$$= x(1 - x) \left( \rho_L - \rho + \eta \frac{1}{K} \left( \frac{\pi}{x} - \frac{1 - \pi}{1 - x} \right) \right). \tag{36}$$

Intuitively, the change in $x$ depends on the difference in subjective discount factors between the two groups as well the difference in the wealth of their newborns. The steady-state is characterized by $\dot{x} = \dot{K} = 0$, which gives, after combining (35) and (36),

$$\rho_L - \rho + (\eta + \tau - g) \left( \frac{\pi}{x} - \frac{1 - \pi}{1 - x} \right) = 0. \tag{37}$$

This is a quadratic equation in $x$ which has one and only one solution $x \in (0, 1)$, which pins down the steady-state wealth share $x$ corresponding to a given value $\rho_L$. Product market clearing requires that
Taking the limit \( \pi \)

wealth in the group of entrepreneurs and rentiers must be zero means that

\[
(\pi p + (1 - x)\rho_L)q = \tau - i.
\]

This equation pins down \( q \) as a function of \( x \). Finally, in steady-state, Equation 2 holds, which also gives the steady-state interest rate \( r \) as a function of \( x \):

\[
r = \pi p + (1 - x)\rho_L + g - \tau.
\]

The next proposition shows that, when \( \pi \) is close enough to zero (i.e., entrepreneurs account for a small share of the total population), changes in \( \rho_L \) can generate the full spectrum of interest rates considered in Proposition 2.

**Proposition 6.** Denote by \( r_\pi(\rho_L) \) the equilibrium interest rate as a function of the subjective discount factor of workers. The following are true:

1. \( r_\pi(\cdot) \) is an increasing function of \( \rho_L \);
2. As \( \pi \) tends to zero, \( r_\pi(\cdot) \) spans the interval \((g - \tau, \rho + \eta)\):

\[
\lim_{\pi \to 0} \lim_{\rho_L \to 0} r_\pi(\rho_L) = g - \tau,
\]

\[
\lim_{\pi \to 0} \lim_{\rho_L \to +\infty} r_\pi(\rho_L) = \rho + \eta;
\]

3. As long as \( \rho < (\tau - i)\frac{\eta - (g - \tau)}{\eta} \), there exists \( \pi \) small enough that the distribution of workers always has a thinner tail than the distribution of entrepreneurs or rentiers. In this case, Proposition 2 gives Pareto inequality \( \theta \) for the full distribution of entrepreneurs, rentiers, and workers.

**Proof of Proposition 6.** Denote by \( x_\pi(\rho_L) \) the steady state share of wealth owned by entrepreneurs and rentiers. Equation (37) implies that \( x_\pi(\cdot) \) is increasing in \( \rho_L \). The fact that the growth of the average wealth in the group of entrepreneurs and rentiers must be zero means that \( r_\pi(\cdot) \) is increasing in \( x_\pi(\cdot) \):

\[
0 = r_\pi(\rho_L) - \rho + \eta \left( \frac{\pi}{x_\pi(\rho_L)} K - 1 \right),
\]

Combining both results gives us that \( r_\pi(\cdot) \) increases in \( \rho_L \).

To prove the second part of the proposition, note that Equation 37 implies the following expression for \( x \) as a function of \( \rho_L \):

\[
x_\pi(\rho_L) = \begin{cases} 
\frac{1}{2} \left(1 + \theta(\rho_L) - \sqrt{(1 + \theta(\rho_L))^2 - 4\theta(\rho_L)\pi}\right) & \text{if } 0 < \rho_L < \rho \\
\pi & \text{if } \rho_L = \rho \\
\frac{1}{2} \left(1 + \theta(\rho_L) + \sqrt{(1 + \theta(\rho_L))^2 - 4\theta(\rho_L)\pi}\right) & \text{if } \rho_L > \rho
\end{cases}
\]

where \( \theta(\rho_L) \equiv (\eta - (g - \tau))/(\rho - \rho_L) \). Taking the limit with respect to \( \rho_L \) gives:

\[
\lim_{\rho_L \to 0} x_\pi(\rho_L) = \frac{1}{2} \left(1 + \theta(0) - \sqrt{(1 + \theta(0))^2 - 4\theta(0)\pi}\right),
\]

\[
\lim_{\rho_L \to +\infty} x_\pi(\rho_L) = 1
\]

Taking the limit \( \pi \to 0 \) gives us

\[
\lim_{\pi \to 0} \lim_{\rho_L \to 0} x_\pi(\rho_L) = 0, \quad \lim_{\pi \to 0} \lim_{\rho_L \to +\infty} x_\pi(\rho_L) = 1.
\]
Therefore, in terms of the interest rate, using (39) and (40), we have that
\[
\lim_{\pi \to 0} \lim_{\rho_L \to 0} r_{\pi}(\rho_L) = g - \tau, \quad \lim_{\pi \to 0} \lim_{\rho_L \to +\infty} r_{\pi}(\rho_L) = \rho + \eta.
\]
To conclude the proof, it remains to show that, if \( \rho < (\tau - i) \frac{\eta - (g - \tau)}{\eta} \), there exists \( \pi \) small enough so that workers never dominate the right tail. To see why, denote \( \theta_{\pi,\rho}(\rho_L) \) to be Pareto inequality for workers as a function of \( \rho_L \). We have:
\[
\theta_{\pi,\rho}(\rho_L) = r_{\pi}(\rho_L) - \rho_L = 1 - \frac{1 - \pi}{1 - x_{\pi}(\rho_L) \eta + \tau - g}.
\]
Since \( x_{\pi}(\cdot) \) is increasing in \( \rho_L \), we have that \( \theta_{\pi,\rho}(\cdot) \) is decreasing in \( \rho_L \), and, therefore, that it is bounded by its limit as \( \rho_L \to 0 \). We can express this upper bound in terms of exogeneous parameters:
\[
\lim_{\rho_L \to 0} \theta_{\pi,\rho}(\rho_L) = \lim_{\rho_L \to 0} \frac{r_{\pi}(\rho_L)}{\eta} = g - \tau + \rho \frac{\sqrt{(1+\pi) - 4\theta(0)\pi} - 1 + \theta(0) - \sqrt{1+\theta(0)^2 - 4\theta(0)\pi}}{\eta},
\]
where the equality line uses (39). On the other hand, as seen in the proof of Proposition 2, Pareto inequality for entrepreneurs/rentiers, \( \max(\theta_{E,\pi}(\cdot), \theta_{R,\pi}(\cdot)) \), reaches its miminum for \( r = r^* \) at \( \frac{\pi - \rho}{\eta + \tau - i} \). Putting the two results together, we get that a sufficient condition for entrepreneurs and rentiers to always dominate the right tail (i.e., for \( \max(\theta_{E,\pi}(\cdot), \theta_{R,\pi}(\cdot)) \) to be higher than \( \theta_{L,\pi} \)) is
\[
\frac{g - i - \rho}{\eta + \tau - i} = g - \tau + \rho \frac{\sqrt{(1+\pi) - 4\theta(0)\pi} - 1 + \theta(0) - \sqrt{1+\theta(0)^2 - 4\theta(0)\pi}}{\eta}.
\]
Taking the limit \( \pi \to 0 \), this inequality converges to:
\[
\frac{g - i - \rho}{\eta + \tau - i} \geq \frac{g - \tau}{\eta} \quad \Leftrightarrow \quad \rho \leq (\tau - i) \frac{\eta - (g - \tau)}{\eta}.
\]
Therefore, as long as \( \rho < (\tau - i) \frac{\eta - (g - \tau)}{\eta} \), there exists \( \pi \) small enough so that (41) is satisfied.

\[\square\]

### B Appendix for Section 3

#### B.1 Sufficient statistic in the rentier regime

So far, we have derived our sufficient statistic under the assumption that we are in the entrepreneur regime. We now show that that the sufficient statistic in words (i.e., Equation 7) also holds in the rentier regime. Rentiers own a diversified portfolio. Their portfolio can be seen as a “representative” tree with payout \( \tau \) (the cash flow due to the fraction of trees that blossom every period) minus \( i \) (the negative cash flow due to the investment in existing trees) that grows at rate \( g - \tau \) (the growth rate of trees that keep growing minus the fraction of trees that blossom). This implies a payout yield \( (\tau - i)/q \) and a growth rate \( g - \tau \). Note that the return on wealth \( r \) can be written as
\[
r = \frac{\tau - i}{q} + \frac{g - \tau}{.}
\]

\[\text{Payout yield} \quad \text{Growth rate of cash flows}\]
Plugging this expression for \( r \) in the expression for Pareto inequality (see Proposition 2), we obtain
\[
\theta = \max \left( \frac{g - \frac{i}{q} - \rho}{\eta + \tau}, \frac{g - \tau + \frac{i}{q} - \rho}{\eta} \right).
\]

Differentiating with respect to \( r \) gives us
\[
\partial_r \log \theta = \begin{cases} 
-\frac{\partial \log q}{r - \rho} & \text{for } r < r^* \text{ (entrepreneur regime)} \\
\frac{g - \frac{i}{q} - \rho}{1 - \frac{1}{q}} & \text{for } r < r^* \text{ (rentier regime)}
\end{cases}
\]

The key takeaway is that the sufficient statistic (7) holds regardless of whether we are in the entrepreneur or rentier regime. In other words, the effect of \( r \) on Pareto inequality can be written as
\[
\partial_r \log \theta = \text{Payout yield} \times \frac{\text{Duration}}{\text{Growth rate of wealth}}, \tag{42}
\]
for households reaching the right tail of the wealth distribution (see Appendix B.5 for a generalization of this insight). The only special thing about rentiers is that, because the “representative tree” they own has a constant growth rate, the numerator in the sufficient statistic is equal to one. Mathematically, this comes from the fact that the payout yield of the representative tree is exactly equal to the inverse of the duration (see Footnote 27). A more intuitive explanation is that, as rentiers earn the required return on wealth \( r \) every period, the growth rate of their wealth increases one-to-one with an increase in the required return on wealth.

### B.2 Extension: stochastic aggregate growth

Suppose that all of the assumptions in the “stochastic aggregate growth” extension in Section 3.2 hold. The instantaneous return of holding a tree is given by
\[
\frac{dR_t}{R_t} = \begin{cases} 
\left( \gamma + g - \frac{i}{q} \right) dt + \sigma dZ_t & \text{if } t < T \\
\frac{1}{q} - 1 & \text{if } t = T
\end{cases}
\]
where \( T \) denotes the stochastic time at which the tree blossoms. The market pricing equation is
\[
\mathbb{E}_t \left[ \frac{d\Lambda_t R_t}{\Lambda_t R_t} \right] = 0.
\]
Applying Ito’s lemma, we obtain
\[
r_f + \kappa \sigma = \gamma + g - \frac{i}{q} + \tau \left( \frac{1}{q} - 1 \right).
\]

The evolution of wealth is therefore given by
\[
\frac{dW_t}{W_t} = \begin{cases} 
\left( \gamma + g - \frac{i}{q} - \rho \right) dt + \sigma dZ_t & \text{if } t < T \\
\frac{1}{q} - 1 & \text{if } t = T \text{,} \\
\left( r_f + \kappa \sigma - \rho \right) dt + \sigma dZ_t & \text{if } t > T
\end{cases}
\]
where \( T \) denotes the stochastic time at which the tree blossoms. Note that the presence of aggregate risk does not change the consumption-to-wealth ratio, \( \rho \), due to logarithmic utility. What matters for wealth inequality is the dynamics of wealth normalized normalized by the average wealth in the economy,
which we denote by $\ddot{W}_t$. Its dynamics is given by

$$d\ddot{W}_t = \frac{dW_t}{W_t} - \frac{dA_t}{A_t} = \begin{cases} 
(g - \frac{i}{q} - \rho) \, dt & \text{if } t < T \\
\frac{1}{q} - 1 & \text{if } t = T \\
(r_f + \kappa \sigma - \gamma - \rho) \, dt & \text{if } t > T 
\end{cases}$$

The equations for asset prices and wealth dynamics correspond exactly to the respective equations in the baseline model after defining $r = r_f + \kappa \sigma - \gamma$. The stylized model can therefore be interpreted as growing and stochastic economy. The only change is that $r$ should be interpreted as the required return net of per-capita growth, and $g$ should be thought of as the growth rate of trees relative to per-capita growth. In particular, the expression for Pareto inequality in the stylized model (see Proposition 2) remains unchanged.

**B.3 Extension: leverage**

We now provide a derivation of Equation 10 in the main text. We have

$$\partial_r \log \theta = \partial_r \log \left( -\frac{i\lambda}{q\lambda} + g - \rho \right) = \frac{d}{\frac{i\lambda}{q\lambda} + g - \rho},$$

where the second equality uses the fact that $g$ and $\rho$ are exogenous parameters.

Consider a small change in $r$ and $r_f$, denoted respectively by $dr$ and $dr_f$. Differentiating (8) and (9) gives us

$$d i_{\lambda} = (\lambda - 1) \, dr_f,$$

$$d \log q_{\lambda} = \lambda_M \, d \log q,$$

where $\lambda_M = \lambda q / q_{\lambda}$ denotes market leverage. Differentiating Equation (2), we obtain

$$d \log q = -|\partial_r \log q| \, dr.$$

In words, this says that the value of $q$ only depends on $r$, not $r_f$. Combining these equations, we obtain

$$d \left( -\frac{i\lambda}{q\lambda} \right) = -\frac{d i_{\lambda}}{q\lambda} + \frac{i\lambda}{q\lambda} \, d \log q_{\lambda},$$

$$= -\frac{\lambda - 1}{q\lambda} \, dr_f - \frac{i\lambda}{q\lambda} \lambda_M |\partial_r \log q| \, dr,$$

$$= -\left( \lambda_M - 1 \right) dr_f - \frac{i\lambda}{q\lambda} \lambda_M |\partial_r \log q| \, dr.$$

**B.4 Extension: financial constraints**

We now derive the sufficient statistic in the case of financial constraints. For the sake of generality, we consider the case with leverage, as in Appendix B.3. This will allow us to contrast our result to the
literature on entrepreneurship, which typically assumes an upper bound on leverage and zero equity issuance.

We assume that entrepreneurs face an lower bound on the equity payout yield. More precisely, following the notations in the extension with leverage, denote $\lambda = g - r_f + \lambda (i'(g) - a - (g - r_f))$ the flow of equity financing as share of book equity, and $q_{\lambda} = B - q(1 - q)$ the market value of equity divided by its book value.

The entrepreneur problem is given by

$$rq = \max_{\delta} \left\{ a - i(g) + gq + \tau(1 - q) \right\},$$

subject to $i_{\delta} \leq B q_{\lambda}$.

The baseline model can be seen as a special case where $B = +\infty$ (i.e., no lower bound on the equity payout yield). At the other extreme, another special case often considered in the literature on entrepreneurship, which typically assumes an upper bound on leverage and zero equity issuance.

When the constraint does not bind (i.e., $\nu = 0$), we obtain $i'(g) = q$, as in the baseline model (see Endogenous investment extension in Section 3.2). In contrast, when the constraint binds (i.e., $\nu > 0$), investment is inefficiently low. Moreover, a decline in the required return $r$ has the effect of relaxing the constraint, through its positive effect on $q$.

Denoting by $\nu/\lambda \geq 0$ the Lagrange multiplier on the financial constraint, the first-order condition for investment is

$$(1 + \nu)(1 + \lambda(i'(g) - 1)) = q_{\lambda}.$$}

When the constraint does not bind (i.e., $\nu = 0$), we obtain $i'(g) = q$, as in the baseline model (see Endogenous investment extension in Section 3.2). In contrast, when the constraint binds (i.e., $\nu > 0$), investment is inefficiently low. Moreover, a decline in the required return $r$ has the effect of relaxing the constraint, through its positive effect on $q$.

We now derive the effect of a joint change in $dr$ and $d\lambda f$ on the sufficient statistic in case the constraint binds (note that, in case the constraint does not bind, we revert back to the baseline sufficient statistic). Differentiating the financial constraint, which is an equality, with respect to $r$ and $r_f$, gives us a formula for the response of capital growth $dg$:

$$(1 - \lambda) d\lambda f - (1 + \lambda (i'(g) - 1)) d\lambda = B q_{\lambda} \lambda_M d \log q$$

$$\Rightarrow d\lambda = (1 + \nu) \left( d\lambda f + \lambda_M \left( -\frac{i_{\lambda}}{q_{\lambda}} \partial_r \log q \right) \right).$$

As in the stylized model (see Equation 6), the effect of $r$ on Pareto inequality is given by the relative change in the growth rate of entrepreneurs:

$$d \log \theta = d \log \left( -\frac{i_{\lambda}}{q_{\lambda}} + g - \rho \right)$$

$$= \frac{d\lambda}{-\frac{i_{\lambda}}{q_{\lambda}} + g - \rho}$$

$$= (1 + \nu) \frac{d\lambda f + \lambda_M \left( -\frac{i_{\lambda}}{q_{\lambda}} \partial_r \log q \right) \theta}{-\frac{i_{\lambda}}{q_{\lambda}} + g - \rho}.$$}

Hence, the key difference with the sufficient statistic with leverage (Appendix B.3) is that the formula for the effect of $r$ on $\log \theta$ is multiplied by $(1 + \nu)$.

Moreover, note that the first-order condition for investment is $1 + \nu = q_{\lambda}/(1 + \lambda(i'(g) - 1))$. Since $1 < i'(g) \leq q$, the multiplier is bounded: $1 < 1 + \nu \leq q_{\lambda}$. The lower bound on the multiplier is attained in the limit $B \to \infty$ (i.e., there are no financial frictions). In contrast, the upper bound on the multiplier is attained in the limit $i'(g) \to 1$ (i.e., there are no adjustment costs). In this case, the financial constraint
is the only force that keeps the growth rate of the firm from being infinite, as in Cagetti and De Nardi (2006) and Moll (2014).

**B.5 Extension: heterogeneous firm dynamics**

First, we provide a proof for Proposition 3, which is based on theoretical results in Beare et al. (2021) and Beare and Toda (2022). As a reminder, in the firm dynamics extension, all agents are born with wealth equal to \( q_i K_0 \), where \( s \sim \psi \). Then, the amount of capital that they own \( K_t \equiv W_t / q_t \) evolves according to

\[
\frac{dK_t}{K_t} = \begin{cases} \mu_s \, dt & \text{if } t < T \\ (r - \rho) \, dt & \text{if } t \geq T, \end{cases}
\]

where \( s_t \) denotes the state of the entrepreneur at time \( t \) and \( \mu_s = \frac{\delta_s \, g_s}{q_t} + g_s - \rho \) and \( T \) is the stochastic time at which the entrepreneur diversifies and becomes a rentier (which arrives at Poisson rate \( \tau \)).

**Proof of Proposition 3.** Denote \( p_{EI}(w) \) the joint density of log wealth \( w = \log(W) \) and productivity state for entrepreneurs (an \( S \times 1 \) vector). Denote \( p_{RI}(w) \) the density of log wealth for rentiers. Moreover, denote \( m_{EI}(\xi), m_{RI}(\xi) \) the corresponding moment generating function for wealth:

\[
m_{EI}(\xi) \equiv \int_{-\infty}^{+\infty} e^{\xi w} p_{EI}(w), \quad m_{RI}(\xi) \equiv \int_{-\infty}^{+\infty} e^{\xi w} p_{RI}(w).
\]

It solves the following system of ODEs:

\[
\begin{align*}
\partial_t m_{EI}(\xi) &= \mathcal{D}(\Psi) \xi (\xi \mathcal{D}(\nu) + T - (\tau + \eta) \mathcal{I}) \mathcal{D}(\Psi)^{-1} m_{EI}(\xi) + \eta \mathcal{D}(\Psi)^{-1} \psi, \\
\partial_t m_{RI}(\xi) &= (\xi (r - \rho) - \eta) m_{RI}(\xi) + \tau 1' m_{EI}(\xi),
\end{align*}
\]

where \( \Psi = (q_1, \ldots, q_S)' \) is the vector of prices (i.e., the solution to the HJB, see Equation 13), \( \psi = (\psi_1, \ldots, \psi_S)' \) is the distribution of firm types at birth, \( \mathcal{D}(\nu) \) is the diagonal matrix with diagonal elements given by the vector \( \nu \), and \( \mathcal{I} \) is the identity matrix. Denote \( p_t(w) \) the overall density of log wealth. We have \( p_t(w) = 1' p_{EI}(w) + p_{RI}(w) \), which implies that

\[
m_t(\xi) = 1' m_{EI}(\xi) + m_{RI}(\xi).
\]

We now focus on characterizing the limit \( \lim_{t \to +\infty} m_t(\xi) \). First, recall that we assumed that there exists at least one state \( s \) such that the rate of capital accumulation is positive (i.e., \( \mu_s > 0 \)), which ensures that there exists a unique \( \theta_E > 0 \) such that \( \rho_D \left( \frac{1}{\theta_E} \mathcal{D}(\nu) + T - \mathcal{I}(\tau + \eta) \right) = 0 \) (see Proposition 2 in Beare and Toda, 2022). This allows us to characterize the limit \( m_t(\xi) \) as time tends to infinity:

\[
\lim_{t \to +\infty} m_t(\xi) = \left( 1 + \frac{\tau}{\eta - \xi(r - \rho)} \right) 1' \mathcal{D}(\Psi)^{-1} ((\tau + \eta) \mathcal{I} - (\xi \mathcal{D}(\nu) + T'))^{-1} \eta \psi
\]

if \( 0 \leq \xi < \min(\frac{\eta}{r - \rho}, \frac{1}{\theta_E}) \), and infinity if \( \xi \geq \min(\frac{\eta}{r - \rho}, \frac{1}{\theta_E}) \). That is, \( \lim_{t \to +\infty} m_t \) has a pole at \( \min(\frac{\eta}{r - \rho}, \frac{1}{\theta_E}) \).

Using Theorem 3.1 in Beare et al. (2021), this implies that the long-run wealth distribution has a right Pareto tail with Pareto inequality given by \( \theta = \max \left( \frac{\tau}{\eta}, \theta_E \right) \). This concludes the proof.

Before we prove Proposition 4, we establish the following Lemma.

**Lemma 7.** Denote \( u, v \) to be the left and right eigenvectors associated with the dominant eigenvalue of the matrix \( \frac{1}{\theta} \mathcal{D}(\nu) + T \), normalized so that \( u'1 = u'v = 1 \). The derivative of Pareto inequality \( \theta \) with respect to the interest
rate is given by:
\[ \partial_r \log \theta = \frac{(u \circ v)' \partial_r \mu}{(u \circ v) \mu}, \]
where \( \circ \) denotes the element-wise multiplication of two vectors.

Proof. Denote \( u(\theta, r), v(\theta, r) \) the left and right eigenvector associated with the dominant eigenvalue of the matrix \( \frac{1}{\theta} D(\mu) + T \), normalized so that \( u' = u' = 1 \). Pareto inequality \( \theta \) is implicitly characterized by the following equation
\[ \left( \frac{1}{\theta} D(\mu) + T \right) v(\theta, r) = (\tau + \eta) v(\theta, r). \]
Differentiating this equation with respect to \( r \), we obtain
\[ \left( \frac{1}{\theta} D(\partial_r \mu) - \frac{1}{\theta^2} D(\mu) \partial_r \theta \right) v + \left( \frac{1}{\theta} D(\mu) + T \right) (\partial_r v + \partial_\theta \nu \partial_r \theta) = (\tau + \eta) (\partial_r v + \partial_\theta \nu \partial_r \theta). \]
Left-multiplying by the left eigenvector \( u \) and re-arranging, we obtain
\[ u' \left( \frac{1}{\theta} D(\partial_r \mu) - \frac{1}{\theta^2} D(\mu) \partial_r \theta \right) v = u' \left( \frac{1}{\theta} D(\mu) + T \right) - u'(\tau + \eta) (\partial_r v + \partial_\theta \nu \partial_r \theta), \]
since \( u \) is the left-eigenvector. Finally, one can show that \( u' D(\mu) v > 0 \) since it corresponds to the derivative of \( \xi \to \rho_D(\xi D(\mu) + T) \) at \( \xi = 1/\theta \), which is convex (see Beare et al., 2021). Therefore, we obtain the following expression:
\[ \partial_r \log \theta = \frac{u' D(\partial_r \mu) v}{u' D(\mu) v}. \]
This concludes the proof. \( \square \)

In other words, the derivative of the Pareto exponent with respect to \( r \) is proportional to the derivative of the rate of the growth rate of wealth \( \partial_r \mu \), averaged across productivity states using the vector \( u \cdot v \). Because \( u \) and \( v \) correspond to the eigenvectors associated with the dominant eigenvalue, they are positive element-wise. As we used the normalization \( u' = 1 \), \( u \cdot v \) is a density on the productivity states.

We now prove Proposition 4, which shows that this density \( u \cdot v \) has a physical interpretation: it corresponds to the density of past states for individuals in the right tail of the wealth distribution. We refer the reader to Lecomte (2007) for results on the physical interpretation of left and right eigenvectors of tilted generators in non-stationary environment.

Proof for Proposition 4. Step 1. Consider a function \( f \) defined on the set of states \( \{1, \ldots, S\} \). For an individual \( i \) in the wealth distribution, denote \( F_i = \int_0^{t_i} f(s_{it}) \, ds \) the cumulative sum of \( f(s_{it}) \) since their birth. Denote \( p_E(w, F) \) the cross-sectional density of productivity state \( s \), log wealth \( w \), and \( F \) for entrepreneurs. Denote \( m_E(\xi, \beta) = \int_F e^F p_E(w, F) \, dF \) the moment generating function of \( F \) and denote \( \tilde{m}_E(\xi, \beta) = \int_F e^{\xi F} m_E(w, \beta) \, dw \) the joint moment generating function of \( F \) and \( w \).

Applying the Laplace transform on the Kolmogorov-Forward equation for the wealth distribution of entrepreneurs gives a closed form solution for \( \tilde{m}_E \):
\[ 0 = D(q) \xi (\beta D(f) + \xi D(\mu) + T' - (\eta + \tau) I) D(q)^{-\xi} \tilde{m}_E(\xi, \beta) + \eta D(q)^{\xi} \psi, \]
(46)
where \( f = (f(s_1), \ldots, f(s_5)) \).

We know that \( \xi \to \rho_D(\xi D(\mu) + T) = \tau + \eta \) has a unique positive root (given by \( \xi = 1/\theta \)). This implies that \( \xi \to \rho_D(\beta D(f) + \xi D(\mu) + T) = \tau + \eta \) has a unique positive solution for \( \beta \) close enough to zero, which we denote \( \xi^*(\beta) \). Given the expression for \( \bar{m}_E \) obtained in (46), this implies that \( \bar{m}_E \) has a pole in \( \xi^*(\beta) \). As shown in Beare et al. (2021), the fact that \( \bar{m}_E \) has a pole in \( \xi^*(\beta) \) implies that \( m_E \) has a right tail with exponent \( \xi^*(\beta) \); that is,

\[
\log m_E(\beta, w) \sim \xi^*(\beta)w \text{ as } w \to +\infty,
\]

where \( m_E \) denote the \( s \) coordinate of the vector \( m_E \).

Step 2. Now, note that the expectation of \( F \) conditional on being in the right tail can be expressed using the derivative of \( \log m_E \) at zero:

\[
\mathbb{E} \left[ \int_0^{T_i} f(s_{it}) \, dt \mid \log W_i = w, s_i = s \right] = \partial_{\beta=0} \mathbb{E} \left[ e^{\beta F} \mid \log W_i = w, s_i = s \right] = \partial_{\beta=0} \int_R e^{\beta F} p_\theta(w, f) \, dF = \partial_{\beta=0} \log m_E(\beta, w).
\]

Combining with (47) gives:

\[
\mathbb{E} \left[ \int_0^{T_i} f(s_{it}) \, dt \mid \log W_i = w \right] \sim \xi^*(0)w \text{ as } w \to +\infty.
\]

Using the same derivation as in Lemma 7 gives:

\[
\xi^*(0) = \frac{(u \cdot v)'f}{(u \cdot v)'\mu}.
\]

Combining the last two equations and dividing by \( w = \log W \) gives:

\[
\lim_{W \to \infty} \mathbb{E} \left[ \frac{\int_0^{T_i} f(s_{it}) \, dt}{\log W} \mid W_i = W \right] = \frac{(u \cdot v)'f}{(u \cdot v)'\mu}.
\]

Applying this formula with \( f = \partial_t \mu \) together with Lemma 7 gives:

\[
\lim_{W \to \infty} \mathbb{E} \left[ \frac{\int_0^{T_i} \partial_t \mu \mu \, dt}{\log W} \mid W_i = W \right] = \partial_t \log \theta.
\]

Using the fact that \( \log(W_i) = \log(K_0) + \int_0^{T_i} \mu \mu \, dt + \log(q_{s_i}) \) and dividing both the numerator and denominator by \( \tau_i \) gives the result. \( \square \)

### B.6 Excess q as the prevent value of rents

In Equation 18, we show that, in the stylized model, Tobin’s \( q \) is equal to one plus the present value of rents. We now show that this is a general result. Consider a firm with a (potentially stochastic) sequence of return on capital \( (\text{rok}_i)_{i \geq 0} \) and growth rate \( (\gamma_t)_{t \geq 0} \). Market pricing imposes that the average return of holding the firm during a small period \( dt \) is \( r \, dt \):

\[
r \, dt \equiv \underbrace{\frac{\text{ro}k_i - \gamma_t}{q_t}}_{\text{Payout yield}} \, dt + \underbrace{\gamma_t \, dt + \mathbb{E}_t \left[ \frac{d\gamma_t}{q_t} \right]}_{\text{Expected growth of market value}}
\]

(48)
Solving forward for $q_t$ and assuming that there is no bubbles (i.e., $\lim_{T \to \infty} e^{-\int_0^T (r-g_s) \, ds} q_T = 0$), we obtain

$$q_t = \mathbb{E}_t \left[ \int_0^\infty e^{-\int_0^t (r-g_{t+s}) \, ds} (r_{t+s} - g_{t+s}) \, ds \right],$$

$$= \mathbb{E}_t \left[ \int_0^\infty e^{-\int_0^t (r-g_{t+s}) \, ds} (r_{t+s} - r_t) \, ds + \int_0^\infty e^{-\int_0^t (r-g_{t+s}) \, ds} (r - g_{t+s}) \, ds \right],$$

$$= \mathbb{E}_t \left[ \int_0^\infty e^{-\int_0^t (r-g_{t+s}) \, ds} (r_{t+s} - r_t) \, ds \right] + 1.$$

This is an application of Feynman-Kac formula. Note that we assume that $r$ is constant over time for simplicity, but the same formula would hold true with a time-varying required return $(r_t)_{t \geq 0}$. Multiplying by $K_t$ gives us

$$(q_t - 1)K_t = \mathbb{E}_t \left[ \int_0^\infty e^{-rs} (r_{t+s} - r)K_{t+s} \, ds \right].$$

Hence, $(q_t - 1)K_t$ can be interpreted as the present value of rents.

C Appendix for Section 4

C.1 Estimating the sufficient statistic

C.1.1 Methodology

**Equity payout yield.** We start by showing that the equity payout yield can be written as the sum of a dividend yield and a buyback yield. Denote by $CF_t \, dt$ the amount of cash distributed to equity holders during a time period $dt$. This cash can be distributed through dividends or though share repurchases. Denoting $D_t$ the dividend per share, $P_t$ the price per share, and $N_t$ the number of outstanding shares, the following accounting identity holds:

$$CF_t = N_t D_t - P_t \, dN_t.$$

Dividing by the market value of the firm equity $N_t P_t$, we obtain

$$\frac{CF_t}{N_t P_t} = \frac{D_t}{P_t} - \frac{dN_t}{N_t}.$$

This says that the equity payout yield is the sum of the dividend yield and the buyback yield. The buyback yield is defined as the cash distributed to shareholders through share repurchases divided the market value of the firm. It is the opposite of the growth in the number of shares (in particular, it is positive when firms repurchase shares and negative if they issue shares).

In Compustat, we observe dividends distributed as $dv$ and the market value of equity as $mkvalt$ (or the number of shares $csho$ times the price per share $prcc_c$ if it is missing). In a given year, we construct the dividend yield as

$$\text{Dividend yield} = \begin{cases} 
\frac{dv}{mkvalt} & \text{post-IPO, if public in 2015} \\
0 & \text{pre-IPO, if public in 2015} \\
0 & \text{if private in 2015}
\end{cases}.$$  \hfill (49)

We then construct the lifetime average dividend yield by taking a simple average over all the years.
since the founding date.

For each firm in Compustat in 2015, we observe the total number of shares in 2015 as \( \text{chso}_{2015} \). To account for any stock split between the date of IPO, \( t_{\text{IPO}} \) and 2015, we adjust it by \( \frac{\text{cumadj}_{t_{\text{IPO}}}}{\text{cumadj}_{2015}} \). We then obtain \( N_{0t} \), the initial number of shares, from the number of shares owned by founders just before the IPO, which is reported in the firm S-1 form—the assumption here is that the founders neither buy or sell their initial shares between the founding date and the IPO. In some cases, the S-1 form specifies that the founders were granted supplementary shares between the funding date and the IPO (as part of their labor compensation) or that they bought supplementary shares during funding rounds—in either case, we subtract these shares from the number of shares held by founders before IPO.

The lifetime average buyback yield is therefore given by:

\[
\text{Lifetime average buyback yield} = \begin{cases} \frac{-\log \left( \frac{\text{chso}_{2015}}{N_{0t}} \right)}{\text{cumadj}_{2015} - t} & \text{if public in 2015} \\ 0 & \text{if private in 2015} \end{cases}.
\] (50)

Said differently, for a firm that is public in 2015, (i) we compute its average buyback yield post-IPO as the opposite of the logarithmic growth in the number of its shares outstanding, while (ii) we compute its average buyback yield pre-IPO as the logarithm of the proportion of shares outstanding owned by founders right after IPO, divided by the age of the firm.\textsuperscript{40}

**Market leverage.** For each firm in Compustat, we compute debt as asset at minus cash che minus shareholder equity seq. We construct market leverage as

\[
\text{Market leverage} = \begin{cases} \frac{\text{mkvalt+ (at-che-seq)}}{\text{mkvalt}} & \text{if public in 2015, post-IPO} \\ \text{average among public firms} & \text{if public in 2015, pre-IPO} \\ \text{average among public firms} & \text{if private in 2015} \end{cases}.
\] (51)

**Growth rate of wealth.** Denote \( W_{2015} \), the wealth of each household normalized by the average wealth in the economy. We obtain it by dividing the wealth reported in the 2015 Forbes list by the aggregate household net worth of households from the financial accounts of the United States in 2015 (TNWBSHNO on FRED), divided by the number of households from the Census (TTLHH on FRED). We then compute the lifetime average growth rate using Equation 23.

## C.1.2 Sensitivity analysis and sampling uncertainty

We now assess the robustness of our estimated sufficient statistic.

**Sensitivity analysis.** Our individual statistics (i.e., lifetime average equity payout yield, market leverage, and growth rate of wealth) might be measured with a bias. To assess the sensitivity of our estimated sufficient statistic to these potential biases, we re-estimate the sufficient statistic after shifting uniformly all individual statistics by a given number. We report the results in Table 7, with bootstrapped 95% confidence intervals.

The first two rows report the sufficient statistic after shifting the equity payout yield of all individuals in our sample by ±0.5 pp. The third and fourth rows report the sufficient statistic after shifting the market leverage by ±0.1. Unsurprisingly, decreasing the equity payout yield or increasing the market leverage both tend to decrease the sufficient statistic \( \partial_r \log \theta \), as they correspond to an increased reliance on external sources of financing. Finally, the fifth and sixth rows report the sufficient statistic after shifting the growth rate of wealth by ±5 pp.

\textsuperscript{40}See Boudoukh et al. (2007) for similar measurements of the buyback yield post-IPO.
Table 7: Sensitivity analysis for $\hat{\partial_r \log \theta}$

<table>
<thead>
<tr>
<th></th>
<th>Estimate</th>
<th>95% Confidence interval</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>p5</td>
<td>p95</td>
</tr>
<tr>
<td>Equity payout yield $-0.5pp$</td>
<td>-5.3</td>
<td>-6.9</td>
</tr>
<tr>
<td>Equity payout yield $+0.5pp$</td>
<td>-3.8</td>
<td>-5.3</td>
</tr>
<tr>
<td>Market leverage $-0.1$</td>
<td>-4.0</td>
<td>-5.4</td>
</tr>
<tr>
<td>Market leverage $+0.1$</td>
<td>-5.1</td>
<td>-6.7</td>
</tr>
<tr>
<td>Growth rate $-5pp$</td>
<td>-5.4</td>
<td>-7.3</td>
</tr>
<tr>
<td>Growth rate $+5pp$</td>
<td>-3.9</td>
<td>-5.2</td>
</tr>
<tr>
<td>Growth rate assuming initial wealth $W_{t0} = 10$</td>
<td>-5.5</td>
<td>-7.3</td>
</tr>
<tr>
<td>Growth rate assuming initial wealth $W_{t0} = 0.1$</td>
<td>-3.9</td>
<td>-5.2</td>
</tr>
</tbody>
</table>

Alternatively, we explore the sensitivity of our estimated sufficient statistic to the lifetime average growth rate of wealth by setting the entrepreneur’s wealth at founding date, $W_{t0}$, to ten times more or less than the average wealth in the economy in (23) (as opposed to one in the baseline). We find that this does not change the sufficient statistic much. The reason is that the terminal wealth of the individuals in our sample is so high that small changes in their initial wealth do not matter much for their lifetime average growth rate. Note that, in all specifications, the lower bound of the confidence interval is well below zero. These results suggest that our estimated sufficient statistic, or at the very least the sign, is robust to potential biases and sampling uncertainty.

**Alternative estimator.** In the main text, we estimated the sufficient statistic as a ratio of two averages: the average effect of required returns on the growth rate of wealth, in the numerator, and the average growth rate of wealth, in the denominator (see Equation (21)). In the presence of firm heterogeneity, however, theory instructs us to compute the sufficient statistic as the average of a ratio computed at the individual level (Equation 15). To examine the difference between these two methods, we now consider an alternative estimator for our sufficient statistic $\hat{\partial_r \log \theta}_{alt}$:

$$\hat{\partial_r \log \theta}_{alt} = \frac{1}{N} \sum_{i=1}^{N} \left(1 + \text{Market leverage}_{i,T} \times \left(\text{Equity payout yield}_{i,T} \times \text{Duration} - 1\right)\right) / \text{Growth rate}_{i,T}. \quad (52)$$

We report this estimate, as well as the bootstrapped 95% confidence interval, in Table 8. We find that the alternative estimate for our sufficient statistic is very close, but a bit lower (in absolute value). Mechanically, this difference means that individuals whose growth rate is particularly sensitive to the required return (i.e., a more negative numerator in 21) tend to have a high growth rate to begin with (i.e., high denominator in 21).

Table 8: Estimates for $\hat{\partial_r \log \theta}_{alt}$

<table>
<thead>
<tr>
<th></th>
<th>Estimate</th>
<th>95% Confidence interval</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>p5</td>
<td>p95</td>
</tr>
<tr>
<td>Duration = 35 years (baseline)</td>
<td>-4.2</td>
<td>-5.5</td>
</tr>
<tr>
<td>Duration = 20 years</td>
<td>-3.9</td>
<td>-5.0</td>
</tr>
<tr>
<td>Duration = 50 years</td>
<td>-4.4</td>
<td>-5.9</td>
</tr>
</tbody>
</table>

**Notes.** The alternative sufficient statistic is constructed using Equation (52). The 95% confidence-interval is constructed as a percentile bootstrap confidence interval using 1000 replications. Data are from Forbes, Compustat, and S-1 filings.
C.1.3 The sufficient statistic in 1985 and second-order effects

So far, we have estimated our sufficient statistic using 2015 as the reference year. Formally, this sufficient statistic answers the following question: in a counterfactual world in which required returns on wealth were a bit higher, by how much lower would Pareto inequality be?

In our empirical application, however, we are interested in the effect of a non-infinitesimal change in the required rate of return (i.e., a 2 pp. decline). In theory, the effect of such a large change in the interest rate can be obtained by integrating our sufficient statistic over the path from \( r_0 = 7\% \) to \( r_1 = 5\% \):

\[
\log \theta(r_1) - \log \theta(r_0) = \int_{r_0}^{r_1} \partial_r \log \theta|_{r=r'} \, dr',
\]

where \( \partial_r \log \theta|_{r=r'} \) denotes the derivative of \( \log \) Pareto inequality with respect to the required rate of return when it is equal to \( r' \). Along this path, the composition of individuals at the top, as well as the extent to which they use external financing, would change. As a result, the sufficient statistic would change. In this sense, using the sufficient statistic using 2015 as a reference year only gives a first-order approximation for the effect of a change in required returns on Pareto inequality. Using the trapezoidal rule to approximate the integral, a second-order approximation is

\[
\log \theta(r_1) - \log \theta(r_0) \approx \frac{1}{2} \left( \partial_r \log \theta|_{r=r_0} + \partial_r \log \theta|_{r=r_1} \right).
\]

To quantify these second-order effects, we now estimate our sufficient statistic using 1985 as the reference year. Assuming that the only difference between 1985 and 2015 is due to changes in required returns, the average of two quantities gives a second-order approximation for the effect of the change in required returns on Pareto inequality. We use the same methodology as before. We start from 1985 Forbes list and we categorize the top 100 individuals into entrepreneurs, rentiers and financiers. Table 9 presents our results. Relative to 2015, we find much fewer financiers, and, as a result, slightly more entrepreneurs and rentiers. Moreover, entrepreneurs are much more likely to own a private firm in 1985, while they were more likely to own a public firm in 2015. This makes it a bit harder to measure the sufficient statistic as we typically know much less about these private firms. Still, we now do our best to construct the sufficient statistic in 1985 using the same methodology as the one discussed in Section 4.41

<table>
<thead>
<tr>
<th>Group</th>
<th>Count</th>
</tr>
</thead>
<tbody>
<tr>
<td>Entrepreneurs</td>
<td>79</td>
</tr>
<tr>
<td>Public corporation</td>
<td>20</td>
</tr>
<tr>
<td>Private corporation</td>
<td>59</td>
</tr>
<tr>
<td>Rentiers</td>
<td>13</td>
</tr>
<tr>
<td>Financiers</td>
<td>8</td>
</tr>
</tbody>
</table>

Notes. “Entrepreneurs” are defined as individuals who are invested in non-financial firms that they (or a family member) founded; “Rentiers” are defined as individuals who are no longer invested in the firm that they (or a family member) founded; “Financiers” are defined as individuals who are invested in a financial firm that they (or a family member) founded.

Because S-1 forms were not available electronically for this time period, we directly use micro-files (via the Columbia Business School library) to compute the number of shares owned by founders at IPO, \( N_{t0} \). We report the results in Table 10. Overall, we find that top entrepreneurs in 1985 were much more reliant on debt financing than equity financing. In particular, we find that the average equity payout yield in 1985 is \(-0.3\%\), which is much higher than the \(-2.4\%\) estimated in 2015. Mechanically, this

41More precisely, we use the exact same methodology as Appendix C.1.1, after substituting 2015 by 1985.
comes from the combination of the fact that (i) there are much fewer public firms in the sample (and we assign a 0% equity payout yield to private firms) and (ii) the public firms in our sample tend to have been less reliant on external equity financing over their lifetime relative to 2015. One reason could be that returns on wealth were higher in 1985, and firms rationally respond by investing less (see Endogenous investment extension in Section 3.2). Another reason may be that the venture capital industry was much less developed then, which means that young firms faced larger frictions in issuing equity.

In contrast, we find that the average market leverage is 1.65, which is higher than the 1.37 in 2015. This is consistent with the overall evolution of leverage in the nonfinancial corporate sector (see Hall, 2001). Finally, we find that their lifetime average growth rates of wealth is lower, which reflects the fact that wealth inequality was lower in 1985.

Table 10: Summary statistics in 1985

<table>
<thead>
<tr>
<th></th>
<th>Obs.</th>
<th>Average</th>
<th>Percentiles</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Min</td>
</tr>
<tr>
<td>Equity payout yield</td>
<td>79</td>
<td>-0.3%</td>
<td>-4.8%</td>
</tr>
<tr>
<td>Dividend Yield</td>
<td>79</td>
<td>0.2%</td>
<td>0.2%</td>
</tr>
<tr>
<td>Buyback Yield</td>
<td>79</td>
<td>-0.5%</td>
<td>-4.9%</td>
</tr>
<tr>
<td>Market leverage</td>
<td>79</td>
<td>1.65</td>
<td>0.94</td>
</tr>
<tr>
<td>Growth rate</td>
<td>79</td>
<td>0.20</td>
<td>0.06</td>
</tr>
</tbody>
</table>

Notes. This table reports the lifetime average dividend yield, buyback yield, market leverage, and growth rate for the top 100 U.S. individuals in 1985. The construction of each variable is detailed in Appendix C.1.1. Data are from Forbes, Compustat, and SEC S-1 filings.

We now use these results to produce an estimate of our sufficient statistic for the reference year 1985. Using a duration of 35 years, we find that the sufficient statistic in 1985 is −4.2, which is a bit lower (in absolute value) than the sufficient statistic of −4.6 estimated using 2015 data. On the one hand, successful entrepreneurs in 1985 are less likely to rely on equity financing. On the other hand, they use a larger amount of debt financing. Moreover, their lifetime average growth rates of wealth tend to be lower, which magnifies the effect of a given percentage point change in the growth rate of their wealth on Pareto inequality.

Table 11 reports the sufficient statistic for the reference years 1985 and 2015 as well as the average of these two numbers. For a second-order approximation, however, we need to take into account the fact that the duration of a firms is decreasing in $r$. In particular, the Gordon growth model implies that the derivative of duration with respect to the required return is equal to minus duration squared. This suggests that an average duration of 35 years in our time sample is consistent with a duration of $35 - 0.5 \times 35^2 \times 0.01 \approx 23$ years in 1985 and $35 + 0.5 \times 35^2 \times 0.01 \approx 47$ years in 2015.

The second row of Table 11 reports the sufficient statistic using these heterogeneous durations for 1985 and 2015. The average of these two numbers, which can be seen as second-order approximation of the effect of interest rates on Pareto inequality, gives −4.8. This is very close to our first-order approximation using only data from 2015, which gave −4.6. Hence, after taking account the heterogeneity in duration at the beginning and at the end of the sample, our second-order approach gives very similar results as our first-order approach.

---

42 Consider a firm with a positive cash flow stream that grows on average at rate $g$. Using a constant required return $r$, Footnote 27 shows that the duration of the firm is $1/(r - g)$. As a result, the derivative of the duration with respect to the interest rate is $-1/(r - g)^2$. 

---

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C.2 Estimating required returns

C.2.1 The required return on business liabilities

We now describe how we estimate the required return on business liabilities (i.e., corporate equities and debts). We use publicly available annual data from the Integrated Macroeconomic Accounts, which combines sectoral data on income and expenditure from the National Accounts with data on financial transactions and holdings from the Financial Accounts. We focus on the corporate nonfinancial sector (i.e., Table S5) and deflate all variables using the “Consumer Price Index for All Urban Consumers”.

**Return definition.** Consider the return associated with a trading strategy that consists of holding all liabilities issued by the corporate sectors and purchasing all new issuances in every year. The realized return of owning the corporate sector between time \( t \) and \( t + 1 \) is given by

\[
    r_{\text{corp},t+1} = \frac{\text{net operating surplus}_{t+1} - \text{net capital formation}_{t+1}}{\text{net liabilities}_t} + \frac{\text{net liabilities}_{t+1} - \text{net liabilities}_t}{\text{net liabilities}_t}.
\]

(55)

Net operating surplus (line item 8) is a measure of net corporate profit (i.e., value-added minus worker compensation and capital depreciation). Net capital formation (line 28) measures capital formation (which includes investments in real estate, equipment, and intellectual property products) net of depreciation. Net operating surplus minus net capital formation thus accounts for all of the cash flows generated by the corporate sector.

Net liabilities measures the market value of debts and equities issued by the corporate sector minus the financial assets held by the corporate sector. One could construct net liabilities directly (i.e., line item 144 minus line item 114), however, this would be biased due to the fact that corporate bonds are reported at book value instead of market value. This has originally been pointed out by Hall (2001), who proposed a methodology to estimate the market value of bonds using yield data and assumptions on the maturity of bonds. Recently, Crouzet and Eberly (2021) proposed an updated version of the methodology, and we use their data for our application.

The first term in (55) corresponds to the payout yield. Corporate cash flows can be used to pay interests, dividends, stock buybacks or debt repurchases. Given the trading strategy that we consider, all of these uses of corporate cash flows have the same economic implication: they represent flows of cash from corporations to households (see Abel et al., 1989 for an early discussion of this idea). The second term accounts for the contribution of the growth in the market value of liabilities.

To map (55) more closely to the model, we define the following variables:

\[
    r_{\text{ok},t+1} = \frac{\text{net operating surplus}_{t+1}}{\text{capital}_t}, \quad \text{(Return on capital)}
\]

\[
    g_{t+1} = \frac{\text{net capital formation}_{t+1}}{\text{capital}_t}, \quad \text{(Net capital formation rate)}
\]

\[
    q_t = \frac{\text{net liabilities}_t}{\text{capital}_t}, \quad \text{(Tobin’s q)}
\]
The realized return defined in (55) can thus be expressed as
\[ r_{corp,t+1} = \frac{rok_{t+1} - g_{t+1}}{q_t} + \frac{q_{t+1} \times \text{capital}_{t+1} - q_t \times \text{capital}_t}{q_t \times \text{capital}_t}, \]
\[ = \frac{rok_{t+1} - g_{t+1}}{q_t} + g_{t+1} + \frac{\text{capital}_{t+1} - (1 + g_{t+1}) \text{capital}_t}{\text{capital}_t} + \frac{\text{capital}_{t+1} \times q_{t+1} - q_t}{q_t}. \]

We call the sum of the last two terms the “revaluation gains”. This term combines the growth in the replacement value of capital and the growth in Tobin’s q (revaluation of net financial assets liabilities relative to the replacement value of capital). In practice, the first term in this sum is mainly driven by the revaluation of real estate prices (real estate capital is reported using market values), and it averages to roughly zero in our sample.

**Required returns.** To obtain a measure of expected returns, we make two assumptions. First, the investment rate and the return on capital are known one period in advance (i.e., \( E_t[g_{t+1}] = g_{t+1} \) and \( E_t[rok_{t+1}] = rok_{t+1} \)). Second, expected revaluation gains are zero. See Campbell (2017) chapter 5.5.2, for an analogous set of assumptions in the context of stock market returns. Combining the definition of realized returns (55) with these two assumptions, we obtain that expected returns can be written as:
\[ E_t[r_{corp,t+1}] = \frac{rok_{t+1} - g_{t+1}}{q_t} + g_{t+1}, \]  
which is directly observable. From now on, we refer to \( E_t[r_{corp,t+1}] \) as the required return on wealth. The idea is that, the value of net liabilities \( q_t \) is such that the expected return on investor’s wealth is equal to their required return.

**Aggregate per-capita growth.** What matters in our sufficient statistic approach is the decline in \( r \) net of aggregate growth per capita. One simple method is to deflate our measure of required returns by the growth rate of capital per capita. This deflated measure of required returns simply corresponds to the payout yield, \( (rok_{t+1} - g_{t+1})/q_t \), plus the rate of population growth (SPPOPGROWUSA in FRED). A second method is to deflate our measure of returns by TFP growth, as constructed in Feenstra, Inklaar and Timmer (2015) (RTFPNAUSA632NRUG in FRED).

**Results.** We plot the our required returns on wealth series in Figure 8. The key observation is that, for both deflators, there is a substantial decline in the required return net of per-capita growth.

Table 12 contains summary statistics on the return on capital, the required return on wealth (deflated or not), and Tobin’s q from 1985 through 2015. Computing the decline as the change in the average in 2015–2020 compared to the average in 1980–1985, we obtain that the change in the required return on corporate sector liabilities is \(-2.7\) pp. Deflating by the growth rate of capital per capita gives a change of \(-2.3\) pp. To be conservative we use a change in required returns of \(-2\) pp. in the main text.

**C.2.2 Required returns on corporate liabilities versus required returns on corporate debt**

To implement our sufficient statistic, we have assumed that the change in the required return on corporate liabilities \( dr \) was equal to the change in the required return on corporate debt \( dr_f \). To test this assumption, we now separately estimate the change in the required return on corporate debt. We then discuss the effect of this estimate on the change in Pareto inequality due to the change in required returns.
Figure 8: Growth of required returns net of aggregate growth

Table 12: Returns and valuations of the U.S. corporate sector

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Return on capital (%)</td>
<td>6.5</td>
<td>7.6</td>
<td>7.6</td>
</tr>
<tr>
<td>Required return on wealth (%)</td>
<td>7.8</td>
<td>6.4</td>
<td>5.1</td>
</tr>
<tr>
<td>Required return on wealth net of capital growth p.c. (%)</td>
<td>6.0</td>
<td>5.0</td>
<td>3.7</td>
</tr>
<tr>
<td>Required return on wealth net of TFP growth (%)</td>
<td>7.5</td>
<td>5.8</td>
<td>4.7</td>
</tr>
<tr>
<td>Tobin’s q</td>
<td>0.8</td>
<td>1.4</td>
<td>1.9</td>
</tr>
</tbody>
</table>

Notes. This table reports moments for the U.S. non financial corporate sector. The construction of each variable is detailed in Appendix C.2. Data are from the Integrated Macroeconomic Accounts, FRED, and Feenstra, Inklaar and Timmer (2015).

Debt issued by the corporate sector can take the form of bonds or bank loans. Assuming away the probability of default does not change over time, we can directly estimate this required return from the interest rate paid by the corporate sector. Figure 9 plots the required returns on debt, using data on corporate bond yields for firms rated AAA and BAA (from Welch and Goyal, 2008) and data on bank interest rate (using the bank lending rate DPRIME from FRED). In both cases, we deflate using the “Consumer Price Index for All Urban Consumers” in order to obtain real returns. We find that both rates have declined substantially over time.

Following Barkai (2020), we construct the required return on corporate debt by averaging the two series, with weights given by the relative quantity of each type of debt according to the Integrated Macroeconomic Accounts. Similarly to the case of the required return on all corporate liabilities, what matters is the interest rate on debt relative to the growth rate of the economy. Figure 10 plots the resulting interest rate using the same deflators as in Figure 8, while Table 13 reports the average of interest rates in different periods.

We obtain that the change in the real interest rate paid by firms -3.1 pp. Deflating by the growth rate of capital per capita gives a change of –2.6 pp. (compared to –2.3 pp. for the change in the required return on all corporate liabilities). Remember that our approach can easily incorporate heterogeneous declines. As shown in the Leverage extension in Section 3.2, the implied change in Pareto inequality in the case of a heterogeneous decline in the interest rate on debt \( dr_f \) and in the required return on all

43More precisely, since we only need to estimate the decline in required returns, we only need to assume that the probability of default does not change over time.
Figure 9: Required return on corporate debt

Notes. Panel (a) plots the evolution of U.S. corporate bond yields. Panel (b) plots the evolution of the bank lending rate. Both are in real terms. Data are from Welch and Goyal (2008) and FRED.

Figure 10: Deflated required return on corporate debt

Table 13: Required return on corporate debt

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Interest on corporate debt (%)</td>
<td>5.9</td>
<td>4.4</td>
<td>2.8</td>
</tr>
<tr>
<td>Interest on corporate debt net of capital growth p.c. (%)</td>
<td>4.1</td>
<td>3.0</td>
<td>1.5</td>
</tr>
<tr>
<td>Interest on corporate debt net of TFP growth (%)</td>
<td>5.5</td>
<td>3.9</td>
<td>2.4</td>
</tr>
</tbody>
</table>

corporate liabilities $dr$ is:

$$\partial_r \log \theta = \frac{dr_f + \lambda_M \left( \frac{-\lambda}{q_{x}} \partial_r \log q |dr - dr_f\right)}{-\frac{\lambda}{q_{x}} + g - \rho}.$$  

This gives, in words:

$$d \log \theta = \frac{1 - \text{Market leverage}}{\text{Growth rate of wealth}} \times dr_f + \frac{\text{Market leverage} \times \text{Equity payout yield} \times \text{Duration}}{\text{Growth rate of wealth}} \times dr$$
Plugging a heterogeneous change in \( dr = -2 \) pp. and \( dr_f = -2.3 \) pp. in this formula gives \( d \log \theta = 9.5 \). This is very close to the \( d \log \theta = 9.1 \) obtained in the main text, which used a 2pp. homogeneous decline in \( dr \) and \( dr_f \). This small difference justifies our focus on considering a homogeneous decline in required returns in the main text.

### C.3 Estimating Pareto inequality

We now describe how we estimate Pareto inequality in the data. First, we use data from Smith et al. (2022), who use an improved version of the capitalization method developed in Saez and Zucman (2016) to construct wealth share estimates in the US. Relative to Saez and Zucman (2016), the methodology allows for more granular return heterogeneity.\(^{44}\) Using this data, we construct three alternative estimates of Pareto inequality using the “top share estimator” defined in Equation 25, with \( p = 0.001\% \), \( p = 0.01\% \), and \( p = 0.1\% \).

As a robustness check, we also construct two alternative sets of estimates for Pareto inequality using Forbes data on the wealthiest 400 individuals. We only use data on their rank in the list and on their stated wealth. First, we use the log-rank estimator proposed by Gabaix and Ibragimov (2011). The idea is to estimate a cross-sectional regression of log wealth on the log rank minus 1/2 and use the slope of this regression as an estimate of the Pareto exponent of the wealth distribution. To get an estimate of Pareto inequality, we simply take the inverse of this coefficient. Second, following Saez (2001), we use the mean-min estimator

\[
\theta = 1 - \frac{E[W|W > W]}{W},
\]

where \( E[W|W > W] \) is the average wealth of households in the Forbes 400 list and \( W \) is the wealth of the last household in list.

Table 14 contains the beginning, average, and end value of the Pareto inequality estimates over our time period of interest (i.e., 1980 to 2020). Taking the log difference between the average value in the last five years of our sample minus the average value in the first five years of our sample, and averaging across the different measures of Pareto inequality, gives an estimate for the rise in Pareto inequality of 22 log points.

<table>
<thead>
<tr>
<th>Table 14: Estimates of Pareto inequality</th>
</tr>
</thead>
<tbody>
<tr>
<td>---------------------------</td>
</tr>
<tr>
<td>Ratio top shares (1%-0.1%)</td>
</tr>
<tr>
<td>Ratio top shares (0.1%-0.01%)</td>
</tr>
<tr>
<td>Ratio top shares (0.01%-0.001%)</td>
</tr>
<tr>
<td>Mean-min (Forbes 400)</td>
</tr>
<tr>
<td>Log-rank (Forbes 400)</td>
</tr>
</tbody>
</table>

Notes. The table reports estimates of Pareto inequality using, successively, the log ratio between the top 0.1% and the top 1%, the log ratio between the top 0.1% and the top 0.01%, the log ratio between the top 0.001% and the top 0.1%, the log ratio between the average wealth in Forbes 400 and the wealth of the last person in Forbes 400, and the slope estimate in a regression of log rank minus 1/2 on log wealth. Data from Smith et al. (2022) and Forbes.

\(^{44}\)The authors summarize the influence of return heterogeneity on estimated top wealth shares as follow: “In terms of top portfolios, we find that accounting for estimated return heterogeneity makes a difference. First, relative to an equal returns approach, we find a larger role for pass-through business wealth and a smaller role for fixed income wealth. Second, the fixed income portfolio share falls and the equity share rises with wealth at the top. Pass-through business and C-corporation equity wealth are the primary sources of wealth at the top. At the very top, C-corporation equity is the largest component, accounting for 53% of top 0.001% wealth, and pass-through business accounts for 22%. In contrast, pensions and housing account for almost all wealth of the bottom 90%. Third, we find that the fixed income portfolio share at the very top remained relatively stable, whereas under equal returns, the fixed income portfolio share increased substantially since 2000.”
C.4 Evidence beyond the top 100

**Private businesses.** We now use data from the 2016 wave of the *Survey of Consumer Finances* (SCF) to quantify the prevalence of entrepreneurs at the top of the wealth distributions (i.e., individuals who founded or acquired a business that they actively manage). Table 15 presents summary statistics. First, notice that entrepreneurs are over-represented at the top. As in Cagetti and De Nardi (2006), we find that wealthier individuals are much more likely to be entrepreneurs. In the full population, 11% of individuals are entrepreneurs while in the top 0.01%, the fraction increases to 66%. Second, the businesses founded by wealthy individuals tend to be pass-through entities, which is consistent with the evidence in Cooper et al. (2016). For instance, 93% of businesses owned by households in the top 0.01% are partnerships or S corporations. This is in a sharp contrast with the fact that roughly two-thirds of entrepreneurs in the top 100 own public firms (i.e., C corporations).

<table>
<thead>
<tr>
<th>Groups (shares)</th>
<th>Total</th>
<th>Top 1%</th>
<th>Top 0.1%</th>
<th>Top 0.01%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Entrepreneurs</td>
<td>0.11</td>
<td>0.43</td>
<td>0.58</td>
<td>0.66</td>
</tr>
<tr>
<td>Sole proprietorship</td>
<td>0.48</td>
<td>0.09</td>
<td>0.06</td>
<td>0.02</td>
</tr>
<tr>
<td>Partnership</td>
<td>0.35</td>
<td>0.60</td>
<td>0.64</td>
<td>0.63</td>
</tr>
<tr>
<td>S corporation</td>
<td>0.11</td>
<td>0.21</td>
<td>0.24</td>
<td>0.30</td>
</tr>
<tr>
<td>Other corporations</td>
<td>0.06</td>
<td>0.11</td>
<td>0.06</td>
<td>0.05</td>
</tr>
<tr>
<td>Others</td>
<td>0.89</td>
<td>0.57</td>
<td>0.42</td>
<td>0.34</td>
</tr>
</tbody>
</table>

The effect of required returns on wealth inequality in our model depends on the extent to which these businesses rely on external financing (through either equity or debt financing). Due to data limitations, we are unable to produce estimates of the equity issuance and leverage of the firms owned by entrepreneurs in the top 1% in the US. However, we now present evidence from Kochen (2022) that firms in high-income countries frequently use external financing.

*Kochen (2022)* harmonizes data for 11 high-income countries (i.e., Austria, Belgium, Denmark, Finland, France, Germany, Italy, Norway, Spain, Sweden, and the United Kingdom) over the 1996–2018 period using the Orbis database. The dataset contains firm-level data on millions of companies, most of which are private. Table 16 summarizes the importance of debt and equity financing. First, notice that firms use a substantial amount of leverage, which amounts (on average) to 1.5 using book values. This is somewhat higher than what we find amongst public firms in our sample (see Table 2). Regarding equity issuances, on average 8% of firms issue equity in a given year and, conditional on doing an equity issuance, it amounts to roughly 18% of book equity. Putting together, this represents a roughly 8% × 18% ≈ 1.5% annual net equity issuance yield (or a −1.5% buyback yield), which is roughly half as much as the firms in our sample (see Table 2).

How would the sufficient statistic change after incorporating these entrepreneurs? On the one hand, the fact that they use less equity financing will tend to decrease the numerator in the sufficient statistic (21). On the other hand, the fact that their lifetime average growth rate is (most likely) smaller than the entrepreneurs in Forbes (i.e., most of them did not become billionaires) will tend to decrease the denominator in the sufficient statistic. As a result, the overall effect of incorporating these less successful entrepreneurs is ambiguous (see Appendix C.1.2 for the sensitivity of the sufficient statistic to the equity issuance and growth rate of entrepreneurs).

**Venture capital backed firms.** Firms backed by venture capitalists (henceforth VCs) are an important part of the US economy. According to Capshare, 10,400 companies received venture funding in
Table 16: Debt and equity financing: evidence from Kochen (2022)

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Leverage</td>
<td>1.5</td>
</tr>
<tr>
<td>Equity issuance</td>
<td>0.08</td>
</tr>
<tr>
<td>Size of equity issuance</td>
<td>0.18</td>
</tr>
</tbody>
</table>

Notes. “Leverage” represents the ratio of total asset (i.e., debt plus book equity) divided by the book value of equity. “Frequency of equity issuance” represents the share of firms that have issued equity in the current year; “Size of equity issuance” represents the ratio of equity issuance to capital, conditional on equity issuance being positive. All averages are weighted by capital and obtained in Appendix table A.3. in Kochen (2022). While the paper reports the average debt-to-capital ratio, we transform this into an estimate of the average book leverage (capital-to-book-equity ratio) as $1/(1 - \text{debt-to-capital ratio})$. Similarly, we report the size of equity issuances as a share of book equity while the paper reports it as a share of capital.

2018.\textsuperscript{45} On average, the ownership share of the founders decreases by roughly 25% every funding round. Since funding rounds tend to happen every 18 month, this corresponds to an annual dilution rate of 16% (i.e., an equity payout yield of $-16\%$, assuming that no dividends are paid out).

Another way to obtain a measure of the average dilution rate is to divide the total equity raised by firms funded by VC to their total market capitalization. Pitchbook estimates that VC-backed companies have a combined market capitalization of around $3$ trillion in 2022 and that they collectively raised $130$ billion that year.\textsuperscript{46} Combining these two figures gives a (market-capitalization weighted) dilution rate of 4.5\% (i.e., an equity payout yield of $-4.5\%$, assuming that no dividends are paid out). The fact that this estimate is lower than the previous figure reflects the fact that the largest dilution happens in early rounds, that is, in companies with smaller market capitalizations.

While the number of VC firms is small relative to the number of households, it is worth noting that many key employees of these firms receive a substantial proportion of their income in the form equity (i.e., equity grants, stock options, etc.). Equity compensation typically leads to concentrated portfolios due to a mix of vesting time, other restrictions on stock sales, and illiquidity (especially pre-IPO). Our notion of “entrepreneurs” in the model can be interpreted as including not only the founder of a firm, but also any individual who invests the majority of their wealth in the firm. In particular, it also includes employees that receive a substantial proportion of their income as equity. Eisfeldt et al. (2019) reports that equity compensation represents almost 45\% of total compensation to high-skilled labor in recent years and that employees working in VC-backed firms account for approximately 2\% of the workforce. Despite the lack of data on the portfolio of such “human capitalists”, we think that many wealthy, high-skilled employees have portfolios with concentrated holdings. We expect these concentrated holdings to be particularly important for firms that are net equity issuers.

D Appendix for Section 5

D.1 Characterization of equilibrium

Firm policy functions. The optimal labor and investment satisfies

\[
\begin{align*}
    w_t &= (1 - a)(K_t / L_t)^{\alpha}, \\
    q_{s,t} &= 1 + \chi(g_{s,t} - g_s),
\end{align*}
\]

where \( q_s \equiv V_s(K)/K \) is Tobin’s q. Notice that while firms \( s = 1, 2 \) choose different growth rates, they choose the same capital to output ratio. This is because their production technology is identical. The

\textsuperscript{45}Capshare is a “web application that helps businesses manage their stock and assets on one organized platform”. Our statistics are taken from their “2018 Private Company Equity Statistics Report”.

\textsuperscript{46}These two statistics are taken from their “2022 Quantitative Perspectives: US Market Insights”.

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solution is

\[
g_{s,t} = \frac{g_s}{\lambda} + \frac{1}{\lambda} (q_{s,t} - 1),
\]

\[
L_{s,t} = (1 - \alpha)^{-\frac{1}{2}} w_t^{-\frac{1}{2}} K_{s,t}.
\]

**Firm valuations.** Using the optimal policy functions as well as the definition \( MPK \equiv F_K(K, L) = \frac{f(K, L) - wL}{K} \), we have that

\[
0 = \left( MPK_t - r_t + \tau (q_{1,t} - 1) - (r_t + \tau - g_0)(q_{0,t} - 1) + \frac{1}{2\chi}(q_{0,t} - 1)^2 \right) dt + \mathbb{E}_t \left[dq_{0,t} \right], \quad (57)
\]

\[
0 = \left( MPK_t - r_t - (r_t - g_1)(q_{1,t} - 1) + \frac{1}{2\chi}(q_{1,t} - 1)^2 \right) dt + \mathbb{E}_t \left[dq_{1,t} \right], \quad (58)
\]

Along a balanced growth path (i.e., \( MPK_t = MPK, r_t = r \)), we have

\[
0 = MPK - r + \tau (q_1 - 1) - (r + \tau - g_0)(q_0 - 1) + \frac{1}{2\chi}(q_0 - 1)^2,
\]

\[
0 = MPK - r - (r - g_1)(q_1 - 1) + \frac{1}{2\chi}(q_1 - 1)^2.
\]

**Implications of labor market clearing.** From the first-order condition for labor, we have that the capital to labor ratio is the same at both types of firms. Using the labor market clearing condition (i.e., \( L_{0,t} + L_{1,t} = 1 - \pi \)), the equilibrium wage and \( MPK \) must be

\[
\bar{w}_i = (1 - \alpha) \left( \frac{K_i}{1 - \pi} \right)^{1-a}, \quad MPK_i = a \left( \frac{K_i}{1 - \pi} \right)^{a-1}.
\]

**Law of motion for capital.** The law of motion for detrended capital by firm type is

\[
dK_{0,t} = (g_{0,t} - \tau - \eta) K_{0,t} dt + \eta \pi K dt, \quad dK_{1,t} = (g_{1,t} - \eta) K_{1,t} dt + \tau \psi K_{0,t} dt.
\]

In steady-state, we have

\[
K_0 = \frac{\eta}{\eta + \tau - g_0} \pi K, \quad K_1 = \frac{\tau \psi}{\eta - g_1} K_0, \quad K = \frac{\eta - g_1 + \tau \psi}{\eta - g_1 + \tau \psi} \pi K,
\]

where \( K \) is aggregate capital and \( g_s \) is the steady-state growth rate of the firm of each type, i.e. \( g_s = \frac{g_s}{\lambda} + \frac{1}{\lambda} (q_s - 1) \).

**Duration of aggregate wealth.** In steady state, the aggregate \( q \) is given by

\[
q = \frac{\eta - g_1}{\eta - g_1 + \tau \psi} q_0 + \left( 1 - \frac{\eta - g_1}{\eta - g_1 + \tau \psi} \right) q_1.
\]

The duration of aggregate wealth is given by

\[
D \equiv -\frac{\partial_r q}{q} = -\frac{\partial_r (q_0 K_0 + q_1 K_1)}{q K} = \frac{q_0 K_0}{q K} D_0 + \frac{q_1 K_1}{q K} D_1,
\]

where \( D_s = -\partial_r q_s / q \) denotes the duration of a firm in state \( s \).
Entrepreneur wealth. Let $e_{py,s,t}$ be the equity payout yield of firm in state $s$ at time $t$:

$$e_{py,s,t} = \frac{r_1 - g_{s,t} + \lambda (\text{MPK}_t - t_s (g_{s,t}) - (r_1 - g_{s,t}))}{q_{s,\lambda,t}},$$

where $q_{s,\lambda,t} \equiv 1 + \lambda (q_{s,t} - 1)$. Denoting by $T$ the (random) time at which the firm matures. Assuming that $1 + \lambda (\psi q_{1,t} - 1) > 0$ (this ensures that the entrepreneur does not default when transitioning from a growth to a mature firm, and it will be satisfied in our calibration), the wealth of an entrepreneur evolves according to:

$$\frac{dW_t}{W_t} = \begin{cases} (e_{py,0,t} + g_{0,t} - \rho_E) dt + \frac{dq_{0,\lambda,t}}{q_{0,\lambda,t}} & \text{if } t < T \\ 1 + \lambda (\psi q_{1,t} - 1) - 1 & \text{if } t = T \\ (e_{py,1,t} + g_{1,t} - \rho_E) dt + \frac{dq_{1,\lambda,t}}{q_{1,\lambda,t}} & \text{if } t > T. \end{cases} \tag{59}$$

Denote by $W_{E,0,t}$ and $W_{E,1,t}$ the detrended total wealth of entrepreneurs in firms of state 0 and 1. The wealth evolves according to:

$$dW_{E,0,t} = \left((e_{py,0,t} + g_{0,t} - \rho_E - \tau - \eta) dt + \frac{dq_{0,\lambda,t}}{q_{0,\lambda,t}} \right) W_{E,0,t} + \eta \pi \frac{K_{0,0,t}}{\lambda} dt,$$

$$dW_{E,1,t} = \left((e_{py,1,t} + g_{1,t} - \rho_E - \eta) dt + \frac{dq_{1,\lambda,t}}{q_{1,\lambda,t}} \right) W_{E,1,t} + \frac{1 + \lambda (\psi q_{1,t} - 1)}{1 + \lambda (q_{0,t} - 1)} W_{E,0,t} dt.$$

Mutual fund wealth. By Walras’ law, labor and product market clearing implies financial market clearing. The mutual fund must therefore hold all wealth not held by entrepreneurs:

$$W_{M,t} = q_tK_t - W_{E,0,t} - W_{E,1,t}.$$

Since the entrepreneurs own levered claims on firms (i.e., levered equity shares), it means that the mutual fund must hold debt. In a steady-state, this is inconsequential, since all assets have the same return. But over a transition path, it means that the revaluation gains of the mutual fund will differ from those of entrepreneurs. We obtain the mutual fund’s revaluation gains as a residual

$$\frac{dq_{M,t}}{q_{M,t}} = \frac{q_tK_t \frac{dq_{0,\lambda,t}}{q_{0,\lambda,t}} - W_{E,0,t} \frac{dq_{0,\lambda,t}}{q_{0,\lambda,t}} - W_{E,1,t} \frac{dq_{1,\lambda,t}}{q_{1,\lambda,t}}}{q_tK_t - W_{E,0,t} - W_{E,1,t}}.$$

Worker wealth. Denote $W_{L,t}$ to be detrended worker wealth. Its law of motion is:

$$dW_{L,t} = \left((r_t - \rho_L - \eta) dt + \frac{dq_{M,t}}{q_{M,t}} - E_t \left[ \frac{dq_{M,t}}{q_{M,t}} \right] \right) W_{L,t} + (1 - \pi) \omega_t dt - \rho_L H_t dt, \tag{60}$$

where $H_t \equiv E_t \left[ \int_0^\infty e^{-\int_0^s r_t ds} \omega_t s ds \right]$ is the human wealth endowment of a worker at time $t$.

Foreigner wealth. The law of motion for detrended foreigner wealth is

$$dW_{F,t} = S_{F,t} dt + \left( \frac{dq_{M,t}}{q_{M,t}} - \eta dt \right) W_{F,t},$$

where $S_{F,t}$ is the flow of savings by foreigners.
Pareto inequality. The formula for steady-state Pareto inequality is almost exactly as in the stylized model (see Section 2):

\[
\theta = \max \left( \frac{\text{eqy}_0 + \delta_0 - \rho E}{\eta + \tau}, \frac{r - \rho E}{\eta} \right)
\]

(61)

The key difference is that, in the stylized model, \(r - \rho E\) corresponds to the return of rentiers (i.e., return on a diversified portfolio). Now, it corresponds to the return of holding a mature firm. Since the return of mature firms is deterministic, it must be \(r\) (both in expectation and ex-post).

D.2 Neoclassical growth model as a special case

We now show that our model nests the neoclassical growth model as a special case where:

1. Capital is fully elastic (\(\chi = 0\)) and there is no firm heterogeneity (\(\psi = 0\));
2. All agents are workers (\(\pi = 1\)) and there is no population renewal (\(\eta = 0\)).

For simplicity, we focus on a closed-economy steady-state equilibrium. Using the parameter restriction (1) and the firm valuation equations (57), we obtain

\[q_0 = 1, \quad \text{MPK} - \tau = r\]

In words, this says that there are no rents in equilibrium (i.e., the cost of capital \(r\) equals the net return on capital), which implies that Tobin’s \(q\) is one. Notice that the parameter \(\tau\) now has the interpretation of a depreciation rate.

Using the parameter restriction (2), we have that existing agents own all of future wages and payments to capital, which means that their total wealth is \(W = Y/r\), where \(Y = K^\alpha L^{1-\alpha}\). Given the log utility assumption, their optimal consumption is \(C = \rho Y / r\). Using the product market clearing condition (i.e., \(C = Y\)), we have that

\[r = \rho L.\]

In words, this means that the required return is equal to the subjective discount factor.

Putting together, we obtain the steady-state allocation in the neoclassical growth model, where the net marginal product of capital is equal to the subjective discount factor (i.e., MPK – \(\tau = \rho L\)). In the calibrated model, we relax (1) in order to generate a wedge between the return on capital and the cost of capital and relax (2) in order to have wealth inequality due to concentrated portfolios.

D.3 Formulas for moments in the model

The formulas for the micro moments are:

- **Equity payout yield**
  \[
  \text{Equity payout yield} = \frac{r_t - \delta_{0,1} + \lambda (\alpha K^{\alpha-1}(1 - \pi)^{1-\alpha} - \delta_0 - (r_t - \delta_{0,1}))}{\bar{q}_{0,\lambda,1}}
  \]

- **Growth rate of wealth**
  \[
  \text{Growth rate of wealth} = \text{Equity payout yield} + \delta_0 - \rho E
  \]

- **Duration**
  \[
  \text{Duration} = \frac{1}{r - \delta_0 + \tau} \left( 1 + \tau \delta_1 \delta_1 q_0 / q_1 \right)
  \]

- **Market leverage**
  \[
  \text{Market leverage} = \lambda q_0 / q_{0,\lambda}
  \]
The formulas for the macro moments are:

\[
\text{Return on capital} = \alpha K^{\alpha - 1}(1 - \tau)^{1 - \alpha} - \text{Depreciation rate},
\]

\[
\text{Depreciation rate} = \frac{K_0}{K}(\rho_0 - \rho_E - \tau(\psi - 1)) + \frac{K_1}{K}(\alpha g_1 - \delta_1),
\]

\[
\text{Pareto inequality} = \frac{\eta + \tau}{\eta + \tau} \theta + \rho_E,
\]

\[
\text{Aggregate duration} = \frac{q_0 K_0}{q K} \frac{1}{r - \rho_E} \left(1 + \tau \psi \frac{q_0}{q_1} \frac{1}{r - \rho_E} \right) + \frac{q_1 K_1}{q K} \frac{1}{r - \rho_E},
\]

\[
\text{NFA to domestic wealth} = \frac{q K - W_{E,1} - W_{E,2} - W_L}{W_{E,1} + W_{E,2} + W_L}.
\]

D.4 Domestic savings glut

We now consider an alternative model experiment where the equilibrium decline in \(r\) is driven by decline in the subjective discount factor of domestic households (i.e., workers and entrepreneurs). The key difference with the baseline model is that the shock to foreign savings \(S_F\) is replaced by a shocks to the subjective discount factors \((\rho_L, \rho_E)\).

**Calibration and model experiment.** The model experiment consists of a steady-state comparative static where we shock the subjective discount factors of both workers and entrepreneurs by a common shifter \(\epsilon\). In a steady-state associated with \(\epsilon\), we thus have that the optimal consumption for an agents of type \(n\) with wealth \(W\)

\[
\varepsilon = (\rho_n + \epsilon)W.
\]

Other than that, all other model parameters are exactly as in the baseline calibration. The path of foreign savings is constant at some value \(S_F\). We choose the value \(S_F\) as being equal to its value in the \(r = 6\%\) steady-state of the baseline model. That way, we match the fact that the NFA to domestic wealth ratio is 5\% (i.e., a targeted moment in the baseline model calibration).

We then choose the values \(\epsilon(7\%)\) and \(\epsilon(5\%)\) so that, in the steady-states associated with a required return of 7\% (the initial steady-state) and 5\% (the terminal steady-state), the product market clears. Denote the discount factor “shock” as \(\Delta \rho \equiv \epsilon(7\%) - \epsilon(5\%)\). That way, the equilibrium 2 pp. decline in the required return in the model experiment is entirely driven by the discount factor shock \(\Delta \rho\). This stands in contrast with the baseline model experiment, where the equilibrium decline in the required return is driven by a foreign savings shock \(\Delta S_F\). Other than that, the two experiments are identical.

**Comparative static.** Totally differentiating the expression for Pareto inequality (61) in steady-state, and assuming that we are in the entrepreneur regime, we have that

\[
d \log \theta = \frac{\partial \log \theta}{\partial r} \, dr + \frac{\partial \log \theta}{\partial \rho_E} \, d\rho_E,
\]

\[
= \frac{\partial (\rho_0 + \rho_E)}{\partial \rho_E} \, dr + \frac{\partial (\rho_0 + \rho_E)}{\partial \rho_E} \, d\rho_E,
\]

\[
= \frac{1 + \lambda \frac{\partial \rho_0}{\partial \rho_0}}{\rho_0 + \rho_E} \, dr + \frac{1 + \lambda \frac{\partial \rho_0}{\partial \rho_0}}{\rho_0 + \rho_E} \, d\rho_E.
\]

In words, this gives us

\[
d \log \theta = 1 + \text{Market leverage} \times (\text{Equity payout yield} \times \text{Duration} - 1) \frac{\partial \log \theta}{\partial \rho_E} \, dr + \frac{1}{\text{Growth rate of wealth}} \, d\rho_E.
\]
D.5 Quantifying the intensive and extensive margins of top wealth share growth

We now decompose the rise in top wealth shares in our model into an intensive and extensive margin. This allows us to measure the relative contribution of the growth rate of existing fortunes, as opposed to the inflow of new fortunes, in the rise in top wealth inequality.

Following Gomez (2019), we now decompose the growth rate of the share of aggregate wealth owned by a top percentile, at each time period, into an intensive and extensive term. The intensive term holds constant the composition of individuals in the top percentile over the period of time: it is defined as the wealth growth of individuals who are initially in the top percentile relative to the growth of the average wealth in the economy. In contrast, the extensive term, which is defined as a residual, accounts for all composition changes in the top percentile. More precisely, in our model, this extensive term is the sum of a positive force—i.e., the flow of successful entrepreneurs in the top percentile (of type 0) who displace the less successful ones (of type 1)—as well as a negative force, population growth.47

Figure 11: Decomposing the growth of the top 0.1% into an intensive and extensive term

Figure 11 plots the (annualized) growth of the top 0.1% wealth share in the baseline model experiment, as well as its decomposition into an intensive and an extensive margin, as discussed above. For the first 5 years, the rise in the intensive term explains most of the rise in the top wealth share. This is because the realized returns of individuals at the top (31) are high relative to the rest of the distribution. This comes from the fact that revaluation gains are particularly high for individuals at the top of the wealth distribution, who tend to own levered positions in high-duration firms. However, as realized returns start declining, the contribution of the intensive term declines. In fact, the intensive term is ultimately lower in the new steady state compared to the initial steady state, as the average return on wealth in this economy is now lower.

The rise in the top 0.1% wealth share is ultimately driven by a rise of the extensive term. This increase in the extensive term reflects the fact that, in a low-rate environment, the most successful entrepreneurs accumulate capital more quickly as they face a lower cost of capital. As shown in Figure 11, this higher inflow of new fortunes in the top 0.1% more than compensates for the lower growth rate of existing fortunes. Overall, these results are consistent with evidence from Gomez (2019), Zheng (2019) and Atkeson and Irie (2020), who argue that an increase in the flow of new fortunes in top percentiles has played a substantial role in the recent rise in U.S. top wealth inequality.

47Gomez (2019) further decomposes the extensive margin as the sum of a positive “between” term, which account for the dispersion in wealth growth within top individuals, and a negative “demography” term, which accounts for demographic changes such as a death and population growth.
Finally, this decomposition is useful to relate our theory to the central idea in Piketty and Zucman (2015), which is that Pareto inequality increases with \( r - g \) (i.e., the required return net of per-capita growth). On the one hand, it is true that a decline in \( r - g \) leads to a decrease in the growth rate of existing fortunes relative to the economy, which tends to push down top wealth inequality. This is captured by the long-run decline in the intensive term in Figure 11. However, what our decomposition shows is that this decline in the intensive term is more than compensated by the increase in the extensive term. In words, the lower growth rate of existing fortunes in a low-rate environment is more than compensated by the larger inflow of new fortunes in the top percentiles.

### D.6 Elastic capital calibrations

#### Calibrations

Table 17 reports the model fit for the three elastic capital extensions (i.e., low-elasticity, medium-elasticity, and high-elasticity).

**Table 17: Targeted moments (elastic capital calibrations)**

<table>
<thead>
<tr>
<th>Moment</th>
<th>Period</th>
<th>Data</th>
<th>Low</th>
<th>Medium</th>
<th>High</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conditional micro moments</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Equity payout yield</td>
<td>1985–2015</td>
<td>-0.024</td>
<td>-0.024</td>
<td>-0.024</td>
<td>-0.024</td>
</tr>
<tr>
<td>Growth rate of wealth</td>
<td>1985–2015</td>
<td>0.31</td>
<td>0.31</td>
<td>0.31</td>
<td>0.31</td>
</tr>
<tr>
<td>Market leverage</td>
<td>1985–2015</td>
<td>1.37</td>
<td>1.37</td>
<td>1.38</td>
<td>1.38</td>
</tr>
<tr>
<td>Duration</td>
<td>1985–2015</td>
<td>35</td>
<td>35</td>
<td>34</td>
<td>33</td>
</tr>
<tr>
<td>Macro moments</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Return on capital</td>
<td>1985</td>
<td>0.070</td>
<td>0.074</td>
<td>0.072</td>
<td>0.073</td>
</tr>
<tr>
<td>Depreciation rate</td>
<td>1985–2015</td>
<td>0.080</td>
<td>0.081</td>
<td>0.081</td>
<td>0.081</td>
</tr>
<tr>
<td>Pareto inequality</td>
<td>1985–2015</td>
<td>0.60</td>
<td>0.59</td>
<td>0.60</td>
<td>0.60</td>
</tr>
<tr>
<td>Aggregate duration</td>
<td>1985–2015</td>
<td>21</td>
<td>21</td>
<td>21</td>
<td>21</td>
</tr>
<tr>
<td>Δ Return on capital</td>
<td>1985–2015</td>
<td>-0.05</td>
<td>-0.05</td>
<td>-0.05</td>
<td>-0.05</td>
</tr>
</tbody>
</table>

\[ (g_0, \bar{g}_1, \tau, \psi, \lambda, \bar{K}, \eta, \rho_L, \rho_E, \chi)_L = (0.383, -0.026, 0.424, 0.509, 2.506, 9.372, 0.110, 0.040, 0.047, 454.4) \]
\[ (g_0, \bar{g}_1, \tau, \psi, \lambda, \bar{K}, \eta, \rho_L, \rho_E, \chi)_M = (0.349, -0.028, 0.398, 0.523, 2.395, 10.380, 0.122, 0.045, 0.021, 176.8) \]
\[ (g_0, \bar{g}_1, \tau, \psi, \lambda, \bar{K}, \eta, \rho_L, \rho_E, \chi)_H = (0.335, -0.030, 0.391, 0.535, 2.388, 10.694, 0.129, 0.047, 0.011, 105.7) \]

**Evidence from investment regressions.** In Section 5.6, we consider three alternative calibrations where we use the parameter \( \chi \)—which governs the degree of investment adjustment costs—to match, respectively, a 0.5, 1, and 1.5 percentage point decline of the return on capital in the model experiment. Table 18 reports model object for four calibrations of the model: the baseline calibration, the three elastic capital calibrations, as well as a “very high elasticity” calibration where we target a 4 percentage points decline of the return on capital.

**Table 18: Model-implied regression coefficients (percentage points)**

<table>
<thead>
<tr>
<th>Calibration</th>
<th>( \Delta r_{ok} )</th>
<th>( \Delta \log \theta )</th>
<th>( 1/\chi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baseline</td>
<td>0.0</td>
<td>10.8</td>
<td>0</td>
</tr>
<tr>
<td>Low elasticity</td>
<td>-0.5</td>
<td>9.6</td>
<td>0.2</td>
</tr>
<tr>
<td>Medium elasticity</td>
<td>-1.0</td>
<td>8.1</td>
<td>0.6</td>
</tr>
<tr>
<td>High elasticity</td>
<td>-1.5</td>
<td>7.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Very high elasticity</td>
<td>-4.0</td>
<td>4.0</td>
<td>3.1</td>
</tr>
</tbody>
</table>
The first column reports the targeted long-run decline of the return on capital (i.e., \( \Delta \text{rok} \)). As discussed earlier, we target values from 0 pp. (in the baseline model) to \(-4\) pp. (in the very high elasticity calibration). The second column reports the long-run increase in (log) Pareto inequality (i.e., \( \Delta \log \theta \)). Notice that the rise in Pareto inequality is monotonically decreasing in the degree of capital elasticity. In the most aggressive calibration (i.e., the very high elasticity calibration), the rise is 4 log points, which is about a third of the rise in the baseline model. The last column reports the inverse of the parameter \( 1/\chi \), which we will use to assess whether our calibrations are consistent with the existing empirical evidence on the sensitivity of firm-level investment to the cost of capital. Recall that, in the model, the following structural relationship holds

\[
g_{s,t} = g_s + \frac{1}{\chi} (q_{s,t} - 1). \tag{62}
\]

If the state \( s \) was observed, we could therefore consistently estimate \( 1/\chi \) by running a regression of the firm-level investment rate \( g \) on \( q \) with a state fixed-effect. Alternatively, if the state \( s \) is sufficiently persistent, the state fixed-effect could be proxied by a firm fixed-effect.

A large empirical literature estimates regression models similar to (62). A recurrent finding is that the relationship between investment rates and \( q \) appears to be very weak. For instance, in his review of the early empirical evidence that uses microdata, Caballero (1999) notes that “Although microeconomic data has improved precision, coefficients on the cost of capital and \( q \) in investment equations have remained embarrassingly small.” To confront the model to the data, we will consider more recent evidence from Peters and Taylor (2017), who find larger coefficients that the earlier literature. In comparison, Table 18 reports values for \( 1/\chi \) that range from 0.2 pp. in the low elasticity calibration to 3.1 pp. in the very high elasticity calibration. Table 2 of Peters and Taylor (2017) reports comparable regression coefficient values ranging from 0.3 pp. (for R&D investment) to 0.6 pp. (for physical investment). Taking these numbers at face value, our baseline calibration (with an implied value of \( 1/\chi = 0 \)) is not too far off and the “medium elasticity” calibration compares favorably to the data.

However, the authors also propose an improved measure of Tobin’s \( q \), which accounts for the presence of intangible capital, and obtain larger regression coefficient values, ranging from 1.3 pp. (for R&D investment) to 2.9 pp. (for physical capital). Those values are an almost an order of magnitude larger, and are closer to our “very high elasticity” calibration. While there is substantial uncertainty on how sensitive firm-level investment is relative to the cost of capital, the existing empirical evidence suggest that it is at best moderately sensitive. To match the evidence from Peters and Taylor (2017) with our model, we would need to accept an implausibly large decline in the return on capital (i.e., of roughly 4 pp.) in response to a 2 pp. decline in the required return. Note, however, that even in the “very high elasticity” calibration, Pareto inequality still increases by 4 log points (compared to 10.8 log points in the baseline model).

### D.7 Solution algorithm

We solve the equilibrium transition dynamics of the model in three steps:

1. We solve for the initial steady-state capital \((K_{0,0}, K_{1,0})\) associated with the required return \( r_0 \);
2. We solve for the evolution of the aggregates \((q_{0,t}, q_{1,t}, K_{0,t}, K_{1,t})_{t \geq 0}\) that is consistent with an initial condition \((K_{0,0}, K_{1,0})\) and a sequence of unanticipated and permanent shocks \((dr_t)_{t \geq 0}\).
3. We back out the sequence of foreign saving shocks \((dS_{F,t})_{t\geq 0}\) that is consistent with product market clearing;

4. We solve for the evolution of the entrepreneur wealth distribution \((p_t(W))_{t\geq 0}\).

### D.7.1 Step 1: Steady-states

**Steady-state solution.** To solve for the steady-state \((q_0, q_1, K_0, K_1)\) given a constant required return \(r_0\), we simply solve the following system of equations using a root-finding numerical algorithm:

\[
0 = -a (K_0 + K_1)^{1-a} (1 - \pi)^{1-a} + (r - \delta_0) - \tau (\psi q_1 - 1) + (r + \tau - \delta_0) (q_0 - 0) - \frac{1}{2\chi} (q_0 - 1)^2,
\]

\[
0 = -a (K_0 + K_1)^{1-a} (1 - \pi)^{1-a} + (r - \delta_1) + (r - \delta_1) (q_1 - 1) - \frac{1}{2\chi} (q_1 - 1)^2,
\]

\[
0 = \left( \frac{1}{\chi} (q_0 - 1) - \eta \right) K_0 + \eta \pi K,
\]

\[
0 = \left( \frac{1}{\chi} (q_1 - 1) - \eta \right) K_1 + \tau \psi K_0.
\]

The first two equations correspond to the HJB equations for the firm problem and the last two equations correspond to the laws of motion for aggregate capital.

### D.7.2 Step 2: Computing the trajectory of aggregates

To solve for the trajectory of aggregates in the presence of a sequence of MIT shocks, we repeatedly solve for perfect foresight transition paths. We first describe how to solve for these paths numerically.

**Perfect foresight transition path.** Consider an arbitrary period \(t\). Suppose that, at the beginning of period \(t\), the economy has capital \((K_{0,t}, K_{1,t})\) and a constant required return \(r_t\) going forward. To solve for the perfect foresight transition dynamics of \((q_{0,t+s}, q_{1,t+s}, K_{0,t+s}, K_{1,t+s})_{s\geq 0}\), we use the method proposed by Achdou et al. (2022). The idea is to assume that the economy will be in steady-state at time \(t + T\), for some large value \(T > 0\). We discretize the time interval \([0, T]\) into equi-spaced intervals \(T = \{0, \Delta t, 2\Delta t, \ldots, T\}\). Let \(n \in T\) denote a point on the grid. The algorithm has two steps.

(a) **Backward step.** First, we solve for the terminal values \((q_{0,t+T}, q_{1,t+T})\) as the steady-state values implied by the required return \(r_t\) (see “Step 1: Steady-states”). Then, given a guess for the path of capital \(\{K_{0,n}, K_{1,n}\}_{n \in S}\), we solve for \((q_{0,n}, q_{1,n})_{n \in T}\) backward using the following recursion

\[
q_{0,n} = \frac{q_{0,n+1} + (\text{MPK}_n - q_0 (g_{0,n+1}) + \tau (\psi q_{1,n+1} - q_{0,n+1})) \Delta t}{1 + (r_{n+1} - g_{0,n+1}) \Delta t},
\]

\[
q_{1,n} = \frac{q_{1,n+1} + (\text{MPK}_n - q_1 (g_{1,n+1}) \Delta t}{1 + (r_{n+1} - g_{1,n+1}) \Delta t},
\]

where \(g_{s,n} = \frac{1}{\chi} (q_{s,n+1} - 1)\) for \(s = 0, 1\).

(b) **Forward step.** Given a guess for \((q_{0,n}, q_{1,n})_{n \in T}\) and initial values \((K_{0,t}, K_{1,t})\), solve for \((K_{0,n}, K_{1,n})_{n \in T}\) forward using the following recursion

\[
K_{0,n+1} = (1 + (g_{0,n+1} - \tau - \eta) \Delta t) K_{0,n} + \eta \pi K \Delta t,
\]

\[
K_{1,n+1} = (1 + (g_{1,n+1} - \eta) \Delta t) K_{1,n} + \tau \psi K_{0,n} \Delta t.
\]

We iterate over both steps until the path \((q_{0,t+s}, q_{1,t+s}, K_{0,t+s}, K_{1,t+s})_{s\geq 0}\) converges.
Sequence of permanent required return MIT shocks. Our model experiment consists of feeding a sequence of permanent MIT shocks to the required return. The idea is that, at every $t$, the required return path gets revised to a constant $r_{t+s} = r_t$ for all $s \geq 0$. Given a required return sequence $(r_t)_{t \geq 0}$, we implement the following algorithm. The goal is to solve for $(q_{0,t}, q_{1,t}, K_{0,t}, K_{1,t})^T_{t=0}$, or $\{q_{0,n}, q_{1,n}, K_{0,n}, K_{1,n}\}_{n \in T}$ in grid notation. The algorithm has two steps.

(a) Initial period ($n = 0$). We solve the steady-state associated with $r_0$ (see “Step 1: Steady-states”) and collect $(K_{0,0}, K_{1,0})$, which we store as $(K_{0,0}, K_{1,0})$.

(b) Subsequent periods ($0 < n \leq T/\Delta t$). We solve the perfect foresight transition path associated with a constant required return $r_n$ and initial state $(K_{0,n}, K_{1,n})$ (see paragraph above titled “Perfect foresight transition path”) and collect the initial values of of $(q_{0,n}, q_{1,n})$ in the perfect foresight transition path, which we store as $(q_{0,n}, q_{1,n})$, and next period’s value for $(K_{0,n}, K_{1,n})$, which we store as $(K_{0,n+1}, K_{1,n+1})$.

We thus obtain a full transition path $\{q_{0,n}, q_{1,n}, K_{0,n}, K_{1,n}\}_{n \in T}$ consistent with the sequence of permanent MIT required return shocks.

D.7.3 Step 3: Computing the trajectory of foreign savings

Once we have the evolution of aggregates, we can compute the path for aggregate wealth $(W_t)_{t \geq 0}$, entrepreneur wealth $(W_{E,t})_{t \geq 0}$, and worker financial wealth $(W_{L,t})_{t \geq 0}$ using the formulas in Section D.1. First, we compute the target foreigner wealth as a residual $W_{F,t} = W_t - W_{E,t} - W_{L,t}$. The goal is to find a sequence of foreigner saving rate $(S_{F,t})_{t \geq 0}$, or $\{S_{F,n}\}_{n \in T}$ in grid notation, such that the law of motion for foreign wealth holds at every period. The idea is that, if the financial market clears, then the product market (30) clears (by Walras law). We thus compute $\{S_{F,n}\}_{n \in T}$ as a residual:

$$S_{F,n} = \frac{W_{F,n+1} - W_{F,n}}{\Delta t} \cdot \frac{q_{M,n+1} - q_{M,n}}{q_{M,n}} + (\eta - r_t)W_{F,n} \quad \forall n \in T.$$

D.7.4 Step 4: Computing the wealth distribution

Finally, we approximate the evolution of the wealth distribution of entrepreneurs $(p(W, t))_{t \geq 0}$ over a finite wealth grid $W$ and time grid $T$. Since the wealth distribution is unbounded (i.e., it has a Pareto tail), we use the “Pareto extrapolation” algorithm proposed by Gouin-Bonenfant and Toda (2022). In short, the idea is to approximate the wealth distribution over a finite grid, while accounting for movements of agents in and out of the grid. The method has been shown to be very accurate and robust to grid choices.