The Dynamics of Inequality Indices

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Inequality Indices

- Researchers often study rise in inequality through inequality indices

  Gini Index, Top Shares, Herfindahl Index, Atkison Index…

- Focusing on inequality indices is useful for two reasons:
  1. Summarize an infinite dimensional object (distribution) into a number
  2. They can act as sufficient statistics (e.g. Herfindahl, CES aggregator)

- Yet, there is gap in relating inequality index to wealth dynamics in micro-data

This Paper

- Develop accounting decomposition for the change in any inequality index

  accounting framework + analytical in continuous-time

- Decomposes rise in inequality index into three terms:
  1. Covariance of wealth change with existing wealth
  2. Average variance of wealth changes
  3. Demographic forces (entry/exit in the economy)

- Decomposition useful for at least two reasons
  1. Clarify distinct drivers behind a rise in inequality
  2. Supplement inequality indices while retaining parsimony

Literature Review

- Generalizes existing accounting decompositions in the literature:

  Average (Melitz and Polanec, 2015)

  Variance (Campbell et al., 2019)

  Average in top percentile (Gomez, 2022).

- Older literature on inequality indices

  Decomposition within/between groups (Bourguignon, 1979, Shorrocks, 1995)

  Ordered Weighted Averages (Mehran, 1976, Yaari, 1988).

Roadmap

1. General Theory
   - Dynamics of inequality index in continuous-time
   - Accounting framework to identify each term in the data
2. Special Cases
   - Herfindhal
   - Average in top percentile
   - Gini index

Inequality Index

- Define a distributional index ν as mapping from Cumulative Distributive Functions to real numbers

- Define a inequality index ν as a distributional index that satisfies two properties:

  (i) Scale Invariance: index does not change when all wealths are scaled by constant

  (ii) Transfer Principle: index decreases if a transfer of $\Delta > 0$ is made from an individual with income x_i to another with income y_i, where $x_i - \Delta > y_i + \Delta$

Influence Function

- For a given distributional index ν, define its Influence Function IF_ν as the directional derivative of ν in the direction of $1_{y \geq x}$:

$$IF_\nu(x; G) \equiv \lim_{\epsilon \downarrow 0} \frac{\nu((1-\epsilon)G + \epsilon 1_{y \geq x}) - \nu(G)}{\epsilon}$$

  $IF_\nu(x; G)$ measures the effect of adding an individual at level x to the CDF G on ν

- Influence functions originally developped for robustness diagnostics (Hampel (1968) and Hampel (1974)) and recently revived by Firpo et al. (2009) "Unconditional Quantile Regressions"

Dynamics of Distributional Indices

- Consider a continuous-time economy with a fixed mass of agents
- Assume that wealth x_{it} of individual i at time t follows a diffusion process:

$$dx_{it} = \mu_t(x_{it})dt + \sigma_t(x_{it})dW_{it}$$

  where W_{it} is idiosyncratic Brownian Motion
- Proposition: Denote G_t the CDF of x_{it} at time t. The dynamics of $\nu(G_t)$ is given by:

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$$\frac{\mathrm{d}\nu(G_t)}{\mathrm{d}t} = \int \frac{\mathrm{E}_t\left[\mathrm{d}IF_\nu(x_{it+\tau})\right]}{\mathrm{d}t} dG_t(x)$$

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$$\frac{\mathrm{d}\nu(G_t)}{\mathrm{d}t} = \underbrace{\int \partial_x IF_\nu(x; G_t)\mu_t(x)dG_t(x)}_{\text{Drift}} + \underbrace{\frac{1}{2}\int \partial_{xx} IF_\nu(x; G_t)\sigma_t(x)^2 dG_t(x)}_{\text{Dispersion}}$$

Heuristic Derivation

- With diffusion process, wealth realization of individual i with initial wealth x is

$$x_{it+\Delta t} \approx \begin{cases} x + \mu_t(x)\Delta t + \sigma_t(x)\sqrt{\Delta t} & \text{with prob } 1/2 \\ x + \mu_t(x)\Delta t - \sigma_t(x)\sqrt{\Delta t} & \text{with prob } 1/2 \end{cases}$$

- Therefore, the average effect of his/her wealth change on ν within Δt is:

$$\begin{aligned} & \mathrm{E}_t \left[IF_\nu \left(x_{it+\Delta t} \right) - IF_\nu(x_{it}) | x_{it} = x \right] \\ & = \frac{1}{2} IF_\nu \left(x + \mu_t(x)\Delta t + \sigma_t(x)\sqrt{\Delta t} \right) + \frac{1}{2} IF_\nu \left(x + \mu_t(x)\Delta t - \sigma_t(x)\sqrt{\Delta t} \right) - IF_\nu(x) \\ & = \partial_x IF_\nu(x_{it})\mu_t(x_{it})\Delta t + \frac{1}{2}\partial_{xx} IF_\nu(x_{it})\sigma_t(x_{it})^2\Delta t + o(\Delta t) \end{aligned}$$

Dynamics of Inequality Indices

- Proposition.
  - Scale Invariance $\Longrightarrow$ individual wealth can be normalized by average wealth
    $\Longrightarrow$ the drift term can be interpreted as a covariance
  - Transfer Principle $\Longrightarrow$ the dispersion term is non-negative

- Decomposition applied to inequality indices has natural interpretation:
  - Drift term captures covariance of wealth and wealth growth (rich pulling ahead / perpetuation of inequality)
  - Dispersion term captures variance of wealth growth (mobility / creation of inequality)

Extensions

1. Heterogeneous types
   In this case, the drift and dispersion terms depend, respectively, on average drift and variance at each wealth level

2. Jumps in wealth
   In this case, the dispersion term depends on all higher-order cumulant of wealth growth (as well as higher-order derivatives of IF)

3. Multivariate distributions (e.g. size-weighted average productivity).
   In this case, the drift and dispersion terms depend, respectively, on the drift vector and volatility matrix

4. Demographic forces (entry and exit in economy)
   In this case, there is third "demography" term, which equals average IF of entering agents − average IF of exiting agents

Accounting Framework

- Assume one observes a set Ω of individuals at time $t = 0$ and $t = 1$
- Denote $\Delta x_{it} = x_{i1} - x_{i0}$ the change in x_{it} between these two periods
- Drift term $\int \partial_x IF_\nu(x; G_t)\mu_t(x)dG_t(x)$ can be estimated as

$$\text{Drift} = \frac{1}{|\Omega|}\sum_{i\in\Omega} \partial_x IF_\nu(x_{i0}; G_0)\Delta x_{it}$$

  where $|\Omega|$ denotes the number of individuals in economy.
- Dispersion term can then be estimated as the residual so that

$$\Delta\nu(G_t) = \text{Drift} + \text{Dispersion}$$

  $\Rightarrow$ We get accounting framework for any index

Example 1: Herfindhal Index

- Herfindhal index is defined as

$$\nu(G) = \int x^2 dG(x)$$

- The influence function is

$$IF_\nu(x; G) = x^2 - \int x^2 dG(x)$$

- Hence the dynamics of the Herfindhal index is:

$$\partial_t \nu(G_t) = \int 2x\mu_t(x) dG_t(x) + \int \sigma_t(s)^2 dG_t(x)$$

- The corresponding accounting decomposition is

$$\Delta\nu(G_t) = \frac{1}{|\Omega|}\sum_{i\in\Omega} 2x_{i0}\Delta x_{it} + \frac{1}{|\Omega|}\sum_{i\in\Omega} (\Delta x_{it})^2$$

Example 2: Ordered Weighted Average

- Ordered Weighted Average (OWA) is a subclass of distributional indices:

$$\nu(G) = \int w(G(x))x dG(x)$$

- $\nu(\cdot)$ satisfies Transfer Principle iff $w(\cdot)$ is increasing function
- Examples:
  - Average wealth in top percentile p: $w(r) = \frac{1_{r \geq 1-p}}{p}$
  - Gini index: $w(r) = 2(r-1)$

Example 2: Ordered Weighted Average

- The influence function of an OWA is

$$IF_{\nu}(x;G) = \int^{x} w(G(y))dy - \int \left(\int^{x'} w(G(y))dy \right) dG(x')$$

- Corollary. The dynamics of an OWA is:

$$\partial_t \nu(G_t) = \int w(G_t(x))\mu_t(x)dG_t(x) + \frac{1}{2}\int w'(G_t(x))G_t'(x)\sigma_t(x)^2 dG_t(x)$$

The corresponding accounting decomposition is:

$$\Delta\nu(G_t) = \frac{1}{|\Omega|}\sum_{i\in\Omega} w(G(x_{i0}))\Delta x_{it} + \frac{1}{|\Omega|}\sum_{i\in\Omega} x_{i1}\Delta w(G(x_{it}))$$

Example 2a: Average Wealth in Top Percentile

- Average wealth in top percentile is OWA with $w(r) = \frac{1_{r \geq 1-p}}{p}$
- Applying Proposition above gives dynamics of average wealth in top p is:

$$\partial_t \nu(G_t) = \frac{1}{p} \int_{q_t} \mu_t(x) dG_t(x) + \frac{1}{2} \frac{G_t'(q_t)}{p} \sigma_t(q_t)^2$$

where q_t denotes the quantile

- The corresponding accounting decomposition is

$$\Delta \nu(G_t) = \frac{1}{p|\Omega|} \sum_{i \in \Omega} \frac{1_{x_{i0} \geq q_0}}{p} \Delta x_{it} + \frac{1}{|\Omega|} \sum_{i \in \Omega} x_{i1} \Delta 1_{x_{it} \geq q_t}$$

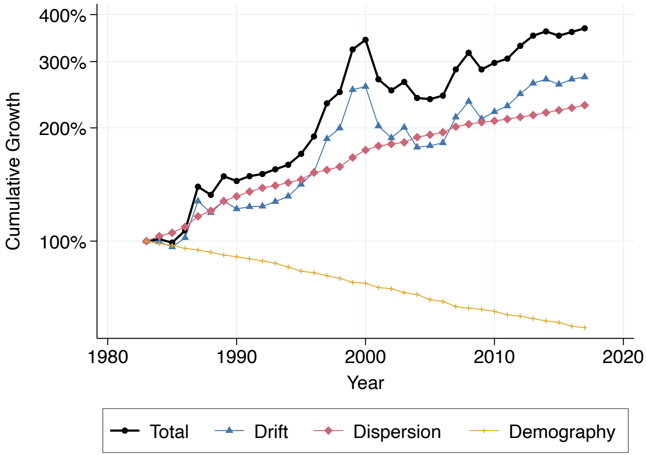

Figure: Decomposing Forbes 400 Wealth Share (Gomez (2022))

Example 2b: Gini Index

- Gini index is OWA with $w(r) = 2r - 1$
- Applying Proposition above gives dynamics of the Gini index as:

$$\partial_t \nu(G_t) = 2G_t(x)\mu_t(x)dG_t(x) + \int G_t'(x)\sigma_t(x)^2 dG_t(x)$$

- The corresponding accounting decomposition is

$$\Delta\nu(G_t) = \frac{1}{|\Omega|}\sum_{i\in\Omega} 2G_0(x_{i0})\Delta x_{it} + \frac{2}{|\Omega|}\sum_{i\in\Omega} x_{i1}\Delta G_t(x_{it})$$

Conclusion

- General approach that unifies & generalizes existing results in the literature

  Melitz and Polanec (2015), Campbell et al. (2019), Gomez (2022)…

- Three distinct drivers of any inequality index
  1. Drift: Covariance of wealth and wealth change
  2. Dispersion: Average variance of wealth change
  3. Demography (not today): Entry/exit of households in the economy

- Measured moments are useful to understand causes of rising inequality + discipline existing models of inequality