The Micro Origins of Aggregate Trends: Drift, Displacement, and Demography

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Abstract

Aggregate statistics such as market concentration, inequality, and productivity are often used to summarize a distribution of firms or households in one number. In this paper, I propose a general framework to relate the change in these statistics to the law of motion of their underlying units. I show that the change in any statistic can be decomposed into three terms: *drift*, equal to the sum of continuing units' changes weighted by their initial marginal effects on the statistic; *displacement*, which captures the effect of these weights changing as units move; and *demography*, which captures the effect of units entering and exiting the economy. Each component can be measured using panel data. This decomposition helps uncover the micro forces behind changes in aggregate statistics, as I illustrate with aggregate productivity, wealth inequality, and market concentration.

KEYWORDS: Distributional statistics, decomposition, panel data, inequality dynamics, aggregate productivity, market concentration, Hulten's theorem, random growth.

Introduction

Empirical researchers routinely summarize distributions with a single number (e.g., a moment, a top share, the Gini coefficient, or the Herfindahl–Hirschman index). One number is easier than a full distribution to track over time, compare across countries or markets, and communicate to policymakers. Such a number can also often serve as a sufficient statistic in economic models (a CES aggregator or a Herfindahl index in models of heterogeneous firms).

When such a statistic moves, the change is usually credited to how units at different positions grow. Inequality rises, the usual story goes, because the rich get richer. This force, which I call drift, is only one of three. Even when rich and poor households expect the same returns, shocks open gaps among households that start with the same wealth, which raises inequality: this is displacement. New households, which start with little wealth, raise it too: this is demography. The point is more general: any nonlinear statistic, from aggregate productivity to market concentration, moves with all three forces, each of which contributes to its trend.

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I first develop an accounting framework that splits the change in any differentiable statistic into these three terms using panel data, without a model of units' dynamics. Drift is the change predicted by a linear approximation of the statistic at the initial positions: the sum of the changes of units present at both dates ("continuing units"), each weighted by the unit's marginal effect on the statistic at the start of the interval. Displacement is what this approximation misses. It arises because the statistic is curved: when units that start at the same position spread apart, the gains of those that rise and the losses of those that fall no longer cancel. Demography is the rest of the change, from units that enter and exit. The framework unifies decompositions that earlier work derived separately for specific statistics, such as the variance, the Gini, and share-weighted productivity.

Displacement has a closed form for common statistics. The simplest case is the variance. Without entry or exit, drift is twice the covariance between units' initial positions and their changes: it is positive when units above the mean gain more than units below it. Displacement is the variance of units' changes, which is never negative: the variance can rise even when units regress toward the mean. For the exponential mean (CES aggregator), displacement is proportional to the relative entropy (or Kullback–Leibler divergence) between units' initial and final weights. For top shares, it is the effect of ranking units by their final rather than initial positions.

The accounting framework holds whatever units' dynamics are. I then ask what each component measures about these dynamics. Suppose that, over a short interval, each unit's position changes by an expected amount plus an idiosyncratic shock, while some units enter and others exit. Then drift comes from units' expected changes, displacement from the variance of their shocks, and demography from entry and exit. All three grow in proportion to the length of the interval. In particular, displacement is not an artifact of observing units at discrete dates: the variance of shocks adds up over time, as for a random walk, so a month contributes about a twelfth of a year's displacement, just as it contributes about a twelfth of a year's drift. The applications confirm this pattern where the interval between observations can be varied. Shocks common to many units, or to a few large ones, show up mainly in drift, which therefore carries most of a statistic's short-run fluctuations, while displacement and demography contribute steadily to its trend.

I also address the practical issues that arise when taking the decomposition to micro data. The main one is measurement error, pervasive in micro data, which biases the split between drift and displacement; evaluating each unit's weight one interval earlier removes this bias.

Finally, to illustrate how the decomposition works in practice, I apply it in three settings that differ in their statistic, their units, and their data: aggregate productivity, wealth inequality in Europe, and the concentration of product markets. In each, the trend is usually credited to drift: productivity growth to firms' average improvement, inequality to the rich getting richer, concentration to larger brands gaining share. I find instead that in productivity and wealth, displacement and demography are of the same order of magnitude as drift. Part of aggregate productivity growth comes from demand reallocating toward the firms whose productivity rose; part of inequality is created among households with the same initial wealth; part of con-

centration comes from market-share instability, brands winning and losing share. Standard measures either attribute displacement to drift, as when a rising top share is taken to mean that the rich grew faster, or omit it, as growth accounting with initial weights does.

The first application is aggregate productivity growth in the United States. Growth accounting weights each firm's productivity growth by its sales at the start of the period, as Hulten's theorem prescribes: this weighted sum is drift. When firms making the same product sell varieties with a constant elasticity of substitution, displacement is the gain from demand reallocating toward the firms whose productivity rose, and firms' sales shares measure it without any estimate of their productivity. Demography is the growth from entering firms, net of those that exit.

Most aggregate productivity growth comes from displacement and demography rather than from firms' average improvement: together they account for about 70 percent of growth, measured from the Solow residual. Most of the slowdown after the 1990s, which Hulten's formula attributes to slower improvement of firms, instead reflects lower displacement and less entry.

Existing growth models also generate all three components, in widely different proportions. Displacement is a fraction of drift in quality-ladder models, where growth comes from frequent incremental improvements, but usually exceeds it in models of idea diffusion, where a few firms make large gains at any time. In Perla et al. (2021), Hulten's formula would even show firms becoming less productive on average while aggregate productivity rises. More generally, for the same average improvement, an economy whose firms pursue riskier R&D, with rare large steps rather than frequent small ones, grows faster.

The second application is wealth inequality in five European countries, measured by the Gini and the top 10% share. In household panels from these countries, poorer households grow faster than richer ones, yet neither statistic moves much on average across countries. Drift, which measures whether existing gaps widen or close, is negative. Two forces offset it: displacement, the creation of inequality among households with the same initial wealth, and demography, as new households start with little wealth. Inferring from this regression toward the mean that inequality falls is Galton's fallacy (Galton, 1886); inferring from the stable statistic that the poor do not catch up is the converse mistake. The components are also new targets for models of the wealth distribution: in a stationary distribution, they sum to zero but need not be zero. Published models split this zero very differently; the panels favor models in which newborns start with no wealth and returns are risky.

The decomposition is useful in three ways. First, it changes how a statistic should be interpreted: any change can come from any of the three components, even over short intervals; a statistic that barely moves can net large offsetting forces. Second, it is a first step in explaining why a statistic moved: it shows which components account for the change and hence which explanations to pursue. The split is accounting rather than causal, but competing explanations often predict different components. Third, it gives models more targets: a model that matches the level or trend of a statistic can still get there through the wrong dynamics, whereas matching the components also checks that it does so for the right reason.

Related literature. The paper builds on random growth models, which relate changes in the density to individual dynamics (Luttmer, 2010; Benhabib et al., 2011; Gabaix et al., 2016; Gomez and Gouin-Bonenfant, 2024). The law of motion of Section 2 (Proposition 2) is the counterpart of their Kolmogorov forward equation for the statistic that researchers track rather than for the whole density: it applies to statistics that are not moments, such as top shares; each of its terms is the limit of a component measured using panel data. Gabaix (2011) shows that idiosyncratic shocks matter for the volatility of aggregates when some firms are large. This paper shows that they also matter for the trends of aggregate statistics whenever the statistic is nonlinear in units' positions, even when every unit is small.

The accounting framework I propose generalizes existing accounting decompositions that separate what this paper calls drift and displacement: Foster et al. (2001) for share-weighted productivity, Gomez (2023) for top shares, Jenkins and Van Kerm (2006) for the Gini, Kambhampati and Kattuman (2009) for the HHI, Hart and Prais (1956) for the variance of log firm size, Campbell et al. (2019) for that of log wealth, and Berman and Bourguignon (2024) for quantiles. Melitz and Polanec (2015) extend the decomposition of Olley and Pakes (1996), which splits share-weighted productivity into an unweighted average and the covariance between size and productivity, to entry and exit; the demography term here compares exiters and entrants with continuing units as they do. Benkard et al. (2026) separate entry, exit, and mergers from the change in the HHI among continuing firms, which they do not split into drift and displacement. Relative to these papers, I extend the decomposition to any differentiable statistic, show analytically and empirically that displacement, which reflects the curvature of the statistic, is first order in time, and propose a systematic way to deal with measurement error, which often biases the split between drift and displacement in micro data.

The paper also relates to the literature on decomposing distributional statistics. An older literature characterizes the indices that can be split across population subgroups or income sources (Bourguignon, 1979; Shorrocks, 1982). A more recent literature asks how a statistic would change if the composition of the population changed (DiNardo et al., 1996; Firpo et al., 2009; Fortin et al., 2011). Firpo et al. (2009) show that, to first order, the change in the statistic is the influence function integrated against the change in composition, which is the form demography takes here. Following the same units over time adds two terms: the slope of the influence function weights units' expected changes (drift), and its curvature weights the variance of their shocks (displacement). In its applications, the paper connects to the literature on market concentration (Grullon et al., 2019; Autor et al., 2020; Covarrubias et al., 2020; Benkard et al., 2026), inequality and mobility (Saez and Zucman, 2016; Genicot and Ray, 2023), growth accounting (Hulten, 1978; Petrin and Levinsohn, 2012; Baqaee and Farhi, 2019, 2020; Aghion et al., 2019; Garcia-Macia et al., 2019), and price indices (Diewert, 1976; Feenstra, 1994; Broda and Weinstein, 2006; Redding and Weinstein, 2020). Baqaee and Farhi (2019) show that Hulten's theorem holds only to first order in productivity changes. Displacement is the corresponding nonlinear term: I show that it is first order in time and that it adds to growth among firms whose varieties are substitutes. The instantaneous mobility measure of Genicot and Ray (2023) is drift; displacement and demography relate this mobility to the change in the statistic

between two cross-sections.

Roadmap. Section 1 presents the accounting identity for panel data, interprets its components, and derives them for common statistics. Section 2 studies its limit in continuous time. Section 3 discusses the practical issues that arise when taking the decomposition to micro data. Section 4 applies the framework to aggregate productivity, wealth inequality in Europe, and product-market concentration.

1 An Accounting Framework

This section traces changes in distributional statistics to how units move, enter, and exit. I give an accounting framework that decomposes observed changes into drift, displacement, and demography using panel data, interpret each component, and derive the components of common statistics.

1.1 The Decomposition for Any Statistic

At date t, unit i has position x_{it} (x_{it} could be a household's wealth or a firm's sales). Let G_t denote the cross-sectional distribution of positions, a measure whose total mass is the number of units. Take a distributional statistic ν that associates to the measure G_t a number $\nu(G_t)$.[1] Over an interval Δt, write $\Delta x_{it} = x_{i,t+\Delta t} - x_{it}$ and $\Delta\nu(G_t) = \nu(G_{t+\Delta t}) - \nu(G_t)$.

The goal of the decomposition is to relate the change in the statistic, $\Delta\nu(G_t)$, to how units move, enter, and exit between t and $t + \Delta t$. Let us first take care of demography, the units that exit or enter. Let C denote the *continuing units*, those present at both dates, and G_t^C and $G_{t+\Delta t}^C$ their distributions at the two dates. I measure the contribution of exit as the change in the statistic when the units that exit are removed at t, $\nu(G_t^C) - \nu(G_t)$, and the contribution of entry as the change when the units that enter are added at $t + \Delta t$, $\nu(G_{t+\Delta t}) - \nu(G_{t+\Delta t}^C)$. As in Melitz and Polanec (2015), exiters are thus compared with continuing units at the start of the interval, entrants with continuing units at its end. The change in the statistic is then the sum of exit, entry, and the change in the statistic of continuing units, $\Delta\nu(G_t^C) = \nu(G_{t+\Delta t}^C) - \nu(G_t^C)$, which comes only from how these units move.

I split this remaining change into a linear part, drift, and the rest, displacement. Drift is the change predicted by a linear approximation of the statistic at the initial positions: the sum of units' changes, each multiplied by the partial derivative of the statistic of continuing units with respect to the unit's position, $\partial\nu(G_t^C)/\partial x_{it}$. I call this derivative the unit's *weight*: it measures how much a marginal increase in that position changes the statistic, holding the other positions fixed. Since the statistic is anonymous, units that start at the same position have the same weight, so only the total change of these units matters for drift, not which of

[1] Formally, ν is a real-valued function on a class of finite measures on $\mathbb{R}$. Since ν depends on units only through G_t, it is anonymous: it is unchanged when two units swap positions. I work with measures rather than probability distributions because entry and exit can affect the size of the population, which matters for statistics that depend on the number of units, such as the Herfindahl–Hirschman index.

them gains and which loses. Displacement is what the linear approximation misses, including the whole effect of the dispersion of outcomes among units that start at the same position: spreading these units around their average change leaves drift unchanged.

Overall, we have decomposed the change from $\nu(G_t)$ to $\nu(G_{t+\Delta t})$ into four steps, each of which represents one term of the accounting framework:

$$\nu(G_t) \xrightarrow[\text{exit}]{} \nu(G_t^C) \xrightarrow[\substack{\text{drift}\\(\text{slope})}]{} \nu(G_t^C) + \sum_{i\in C} \frac{\partial \nu(G_t^C)}{\partial x_{it}} \Delta x_{it} \xrightarrow[\substack{\text{displacement}\\(\text{curvature})}]{} \nu(G_{t+\Delta t}^C) \xrightarrow[\text{entry}]{} \nu(G_{t+\Delta t}).$$

Each component of the decomposition is the change along one arrow, so the components add up to the change in the statistic.

Proposition 1 (Accounting Decomposition)**.** *Suppose the partial derivatives of $\nu(G_t^C)$ with respect to continuing units' positions exist. Then*

$$\Delta\nu(G_t) = \text{Drift} + \text{Displacement} + \text{Demography},$$

where

$$\text{Drift} \equiv \sum_{i\in C} \frac{\partial \nu(G_t^C)}{\partial x_{it}} \Delta x_{it}, \qquad \text{Displacement} \equiv \Delta\nu(G_t^C) - \sum_{i\in C} \frac{\partial \nu(G_t^C)}{\partial x_{it}} \Delta x_{it},$$

$$\text{Exit} \equiv \nu(G_t^C) - \nu(G_t), \qquad \text{Entry} \equiv \nu(G_{t+\Delta t}) - \nu(G_{t+\Delta t}^C),$$

and Demography *is the sum of* Exit *and* Entry*.*

Drift. Drift is the linear part of the change in the statistic of continuing units. We can always further decompose drift into the sum of weights times the average change plus a covariance term between weights and units' changes:

$$\text{Drift} = \sum_{i\in C} \frac{\partial \nu(G_t^C)}{\partial x_{it}} \Delta x_{it} = \left(\sum_{i\in C} \frac{\partial \nu(G_t^C)}{\partial x_{it}} \right) \mathrm{E}[\Delta x_{it}] + \sum_{i\in C} \frac{\partial \nu(G_t^C)}{\partial x_{it}} \left(\Delta x_{it} - \mathrm{E}[\Delta x_{it}] \right), \tag{1}$$

where $\mathrm{E}[\Delta x_{it}]$ is the average change across continuing units. If the statistic is *translation invariant* (i.e., adding the same amount to every position leaves it unchanged), its weights sum to zero, so the first term vanishes and drift is simply the covariance between weights and units' changes.[2] Drift then raises the statistic when the units with the largest weights gain the most.

Displacement. Displacement is the part of the change in the statistic of continuing units that the linear approximation does not capture. Although the accounting framework constructs displacement as a residual, a Taylor expansion expresses it in terms of units' changes. Drift is

[2]One alternative decomposition for the drift term would be in terms of growth rates; that is

$$\text{Drift} = \left(\sum_{i\in C} x_{it} \frac{\partial \nu(G_t^C)}{\partial x_{it}} \right) \mathrm{E}\left[\frac{\Delta x_{it}}{x_{it}} \right] + \sum_{i\in C} x_{it} \frac{\partial \nu(G_t^C)}{\partial x_{it}} \left(\frac{\Delta x_{it}}{x_{it}} - \mathrm{E}\left[\frac{\Delta x_{it}}{x_{it}} \right] \right).$$

The first term is zero when the statistic is *scale invariant*; that is, multiplying every position by the same factor leaves it unchanged.

the first-order term of the Taylor series of the statistic of continuing units around their initial positions; displacement is the sum of the higher-order terms:[3]

$$\text{Displacement} = \sum_{n\geq 2} \frac{1}{n!} \sum_{i_1,\ldots,i_n \in C} \frac{\partial^n \nu(G_t^C)}{\partial x_{i_1 t} \cdots \partial x_{i_n t}} \Delta x_{i_1 t} \cdots \Delta x_{i_n t}. \tag{2}$$

Displacement is therefore zero only for statistics that are linear in positions, such as the average. It depends not only on powers of each unit's change but also on cross products of different units' changes. For a convex statistic, the statistic at the final positions lies above its tangent at the initial positions, so displacement, which is the gap between them, is non-negative. The opposite holds for concave statistics.

To second order in units' changes, displacement takes a simple form, which I call the *trapezoid approximation*. It requires only units' weights at the two dates, not the second derivatives of the statistic:

$$\text{Displacement} \approx \frac{1}{2} \sum_{i \in C} \left(\frac{\partial \nu(G_{t+\Delta t}^C)}{\partial x_{i,t+\Delta t}} - \frac{\partial \nu(G_t^C)}{\partial x_{it}} \right) \Delta x_{it}.$$

To this order, the change in the statistic is the sum of units' changes weighted by the average of their initial and final weights (the trapezoid rule). Drift uses only initial weights, so displacement is half the sum of the changes in weights times units' changes. It can thus be interpreted as the effect of the weights changing as units move. When weights rise for the units that gain and fall for those that lose, each term of the sum is positive.

Demography. Demography is the sum of exit and entry, each of which compares the statistic of all units with that of continuing units at the same date. Exit and entry can each be further split into two counterfactual steps. Exit first shrinks the population to the size of continuing units, removing a random sample of incumbents: this is *mass*, a change in size that leaves the shape of the distribution unchanged. The rest is *selection*, the effect of which units actually exit; it is zero when exiters are distributed like continuing units. With N_t the number of units at t and N^C the number of continuing units,

$$\text{Exit} = \underbrace{\left[\nu\left(\frac{N^C}{N_t} G_t \right) - \nu(G_t) \right]}_{\text{Mass}} + \underbrace{\left[\nu(G_t^C) - \nu\left(\frac{N^C}{N_t} G_t \right) \right]}_{\text{Selection}}.$$

Entry works the same way: entrants distributed like continuing units first enlarge the population (mass), then take their actual positions (selection). If the statistic is *population invariant* (i.e., adding units distributed like the existing ones leaves it unchanged, $\nu(\lambda G) = \nu(G)$ for all $\lambda > 0$), mass is zero, so demography is selection alone (Appendix E).

[3] The series assumes a smooth statistic. Statistics with kinks, such as top shares, have no such series, but Proposition 1 still applies to them: Table 1 gives their displacement directly.

1.2 Drift and Displacement for Common Statistics

Proposition 1 holds for any differentiable statistic. I now report drift and displacement for common statistics, abstracting from entry and exit. Take first the simplest statistic, the average, $\mathrm{E}[x_{it}]$. Every unit has the same weight, $1/N_t$: raising one unit's position by one raises the average by $1/N_t$, whatever the unit and wherever it stands. Drift, the sum of units' changes times their weights, is therefore the average change. Displacement is zero, because the weights do not depend on positions, so they stay the same however units move. The decomposition is

$$\Delta \mathrm{E}[x_{it}] = \underbrace{\mathrm{E}[\Delta x_{it}]}_{\text{Drift}} + \underbrace{0.}_{\text{Displacement}}$$

Consider next the variance, $\mathrm{Var}(x_{it})$. A unit's weight is now proportional to its distance from the mean: raising a unit above the mean raises the variance; raising a unit below the mean lowers it, since the unit moves closer to the mean. Drift is therefore twice the covariance between units' initial positions and their changes. The decomposition is

$$\Delta \mathrm{Var}(x_{it}) = \underbrace{2\mathrm{Cov}(x_{it}, \Delta x_{it})}_{\text{Drift}} + \underbrace{\mathrm{Var}(\Delta x_{it})}_{\text{Displacement}}.$$

Drift captures whether existing gaps widen or close: it is positive when units above the mean gain more than units below it, negative when they gain less, as when units regress toward the mean. Displacement captures new gaps. It is the variance of units' changes: units that start at the same position end up apart when their changes differ. Drift cannot see these new gaps, since units that start at the same position have the same weight: the gains of those that rise cancel the losses of those that fall. Displacement is never negative, so the variance can rise even when units regress toward the mean.

We can proceed in the same way for any other statistic: each unit's weight, $\partial \nu(G_t^C)/\partial x_{it}$, gives drift; comparing drift with the change in the statistic then gives displacement. Although the general decomposition defines displacement as a residual, this residual turns out to take simple, intuitive forms. Table 1 collects drift and displacement for these and other common statistics; Appendix E describes each row in more detail and gives its demography. The exponential mean is the log of a power mean of $e^{x_{it}}$, the form of CES aggregators. Its weights ω_{it} are proportional to $e^{\gamma x_{it}}$; its displacement is the relative entropy between initial and final weights, divided by γ. Ordered weighted averages weight positions by a function of their percentile rank (quantiles, top shares, and the Gini). A unit's weight is its rank weight, so displacement is the change in rank weights valued at final positions: it is zero when no unit changes rank. Finally, the table has a row for the average product of two variables, $\mathrm{E}[x_{it}y_{it}]$. Each unit then has one weight per variable; displacement is the average cross product of the changes in the two variables. Applied to share-weighted productivity, with shares as the second variable, this split corresponds to the decomposition of Foster et al. (2001): their within and between terms sum to drift and their cross term is displacement (Appendices B.1 and E.7).

Overall, the accounting framework unifies decompositions that the literature has discov-

Table 1: Drift and Displacement for Common Statistics

Cross-sectional statistic		Decomposition of $\Delta\nu$	
Name	ν	Drift	Displacement
Average	$\mathrm{E}[x_{it}]$	$\mathrm{E}[\Delta x_{it}]$	0
Variance	$\mathrm{Var}(x_{it})$	$\mathrm{Cov}(2x_{it}, \Delta x_{it})$	$\mathrm{Var}(\Delta x_{it})$
Average product	$\mathrm{E}[x_{it}y_{it}]$	$\mathrm{E}[y_{it}\Delta x_{it}] + \mathrm{E}[x_{it}\Delta y_{it}]$	$\mathrm{E}[\Delta x_{it}\Delta y_{it}]$
Average of a function	$\mathrm{E}[f(x_{it})]$	$\mathrm{E}[f'(x_{it})\Delta x_{it}]$	$\mathrm{E}[\Delta f(x_{it}) - f'(x_{it})\Delta x_{it}]$
Exponential mean ($\omega_{it} \equiv e^{\gamma x_{it}}/\mathrm{E}[e^{\gamma x_{it}}]$)	$\frac{1}{\gamma}\ln \mathrm{E}[e^{\gamma x_{it}}]$	$\mathrm{E}[\omega_{it}\Delta x_{it}]$	$\frac{1}{\gamma}\mathrm{E}\left[\omega_{it}\ln\frac{\omega_{it}}{\omega_{i,t+\Delta t}}\right]$
Ordered weighted average	$\mathrm{E}[w(R_{it})x_{it}]$	$\mathrm{E}[w(R_{it})\Delta x_{it}]$	$\mathrm{E}[x_{i,t+\Delta t}\Delta w(R_{it})]$
Top p average	$\mathrm{E}[x_{it} \mid R_{it} \geq 1-p]$	$\mathrm{E}[\Delta x_{it} \mid R_{it} \geq 1-p]$	$\frac{1}{p}\mathrm{E}[x_{i,t+\Delta t}\Delta \mathbf{1}_{R_{it}\geq 1-p}]$
Gini	$\mathrm{E}[(2R_{it}-1)x_{it}]$	$\mathrm{E}[(2R_{it}-1)\Delta x_{it}]$	$\mathrm{E}[2x_{i,t+\Delta t}\Delta R_{it}]$

Notes: Each row splits the change in the statistic into drift and displacement, assuming no entry or exit (with entry and exit, the same formulas apply to continuing units). $\mathrm{E}[\cdot]$ is the average across units, $\mathrm{Var}(\cdot)$ and $\mathrm{Cov}(\cdot,\cdot)$ the variance and covariance across them, and R_{it} the percentile rank. The top p average and Gini rows use absolute positions, which gives the absolute Gini; replacing x_{it} by the normalized position $\tilde{x}_{it} \equiv x_{it}/\mathrm{E}[x_{it}]$ would give a decomposition for the top p *share*, divided by p, and the *relative* Gini. Appendix E decomposes more statistics, gives demography, and compares the decomposition with existing accounting frameworks (Table E1).

ered separately for specific statistics. It also extends them to any differentiable statistic and proposes a systematic way to treat population turnover for every statistic (Table E1 compares the existing decompositions with the three components).

2 The Components in Continuous Time

The decomposition developed above is an accounting identity, and one could simply take it to the data to measure each component. Before doing so, this section answers three questions. First, why split the change in the statistic this way rather than another? Second, and relatedly, what feature of units' dynamics does each component capture? Third, is displacement, which is second order in units' changes, only an artifact of observing units at discrete dates?

It answers them by deriving the law of motion of distributional statistics from that of individual units. This law of motion is the sum of three terms, the continuous-time limits of the three components of the accounting framework. Section 2.1 treats units that follow diffusions; Section 2.2 extends the result to jumps and to common and granular shocks, and compares it with the accounting over an interval.

2.1 When Units Follow Diffusions

I consider an economy with a continuum of units, which evolve as follows.

Assumption 1 (Unit Dynamics)**.** Unit paths follow a diffusion process: over a short interval

Δt, a unit's position changes by

$$\Delta x_{it} = \mu_t(x_{it})\Delta t + \sigma_t(x_{it})\sqrt{\Delta t} \cdot \varepsilon_{it},$$

where ε_{it} is a mean-zero, unit-variance shock with finite fourth moment, independent of x_{it}, across units, and over time.[4] Over the same interval, units exit with measure $\Psi_t^- \Delta t$ while units enter with measure $\Psi_t^+ \Delta t$.

The assumption says that units' changes are random variables whose mean and variance are proportional to the interval. Per unit of time, the mean of changes among units at position x is $\mu_t(x)$ and their variance is $\sigma_t(x)^2$. Both can depend on position and time in any way, which allows mean reversion, growth that varies with size, and risk that varies with position. Units at the same position can also have different dynamics, as a young firm and an old firm with the same sales do: the results below then hold with $\mu_t(x)$ and $\sigma_t(x)^2$ the averages among units at x (Gyöngy, 1986). Still, the assumption imposes two substantive restrictions. First, the size of shocks shrinks with the interval, which rules out jumps. Second, shocks are independent across units. Section 2.2 relaxes both.

The next proposition derives how the statistic evolves under this law of motion. It requires one more concept, the statistic's *influence function*: the effect of adding a small mass at position x, per unit of mass added. The partial derivatives of Section 1 do not suffice: with a continuum of units, moving a single unit does not change the statistic; nor do they describe entry and exit.

Definition 1 (Influence Function)**.** For a distributional statistic ν, the *influence function* is the Gateaux (directional) derivative of ν from G in the direction of a point mass at x:[5]

$$\mathrm{IF}_\nu(x; G) \equiv \lim_{h \downarrow 0} \frac{\nu\left(G + h\delta_x\right) - \nu(G)}{h},$$

where δ_x denotes a point mass at x.

The slope of the influence function in x is a unit's weight in Section 1, $\partial \nu(G_t^C)/\partial x_{it} = \partial_x \mathrm{IF}_\nu(x_{it}; G_t^C)$, which gives the effect of moving units; its level gives the effect of adding and removing them.

Proposition 2 (Local Law of Motion of a Distributional Statistic)**.** *Under Assumption 1, and assuming that the statistic or the distribution of positions is smooth*[6]*, the statistic changes at the rate*

$$\lim_{\Delta t \to 0} \frac{\Delta \nu}{\Delta t} = \underbrace{\int \partial_x \mathrm{IF}_\nu(x; G_t)\mu_t(x)\,\mathrm{d}G_t}_{\text{Drift}} + \underbrace{\frac{1}{2}\int \partial_{xx} \mathrm{IF}_\nu(x; G_t)\sigma_t(x)^2\,\mathrm{d}G_t}_{\text{Displacement}} + \underbrace{\int \mathrm{IF}_\nu(x; G_t)(\mathrm{d}\Psi_t^+ - \mathrm{d}\Psi_t^-)}_{\text{Demography}}.$$

[4] Equivalently, in continuous time, positions follow the diffusion $\mathrm{d}x_{it} = \mu_t(x_{it})\,\mathrm{d}t + \sigma_t(x_{it})\,\mathrm{d}W_{it}$, where the W_{it} are standard Brownian motions, independent across units.

[5] Hampel (1974) introduced influence functions to assess the sensitivity of statistics to individual observations. The standard definition perturbs G to $(1-h)G + h\delta_x = G + h(\delta_x - G)$, which keeps total mass fixed when G has mass one. Its influence function is $\mathrm{IF}_\nu(x; G) - \int \mathrm{IF}_\nu\,\mathrm{d}G$, so the two definitions coincide when ν is population invariant ($\int \mathrm{IF}_\nu\,\mathrm{d}G = 0$). Adding mass also captures the effect of entry and exit through the size of the population.

[6] Formally, Assumptions A1, A2, and A3 in Appendix A hold. Assumption A2 has two versions: Assumption A2(a) requires a smooth influence function and covers discrete distributions; Assumption A2(b) requires smooth densities and covers influence functions with kinks or jumps.

Moreover, in the population, the drift, displacement, and demography of Proposition 1, with each unit's weight given by the slope of the influence function at its position, converge to the respective terms above once divided by Δt.[7]

I now provide a heuristic derivation of this result (Figure 1). Consider the units at position x over a short interval. Their expected change $\mu_t(x)$ moves the statistic through the slope of the influence function: this is drift (panel (a)). Their shocks spread them apart in both directions: the linear effects of the shocks cancel, but their curvature effects add, so displacement weights the variance of shocks $\sigma_t(x)^2$ by half the curvature of the influence function (panel (b)). Entry at x adds $\mathrm{IF}_\nu(x; G_t)$ per unit of mass, while exit subtracts it, so demography is the influence function integrated against entrants minus the same integral against exiters (panel (c)). Aggregating these effects over the distribution and dividing by the interval gives the proposition, which Appendix A proves formally.

The proposition has three implications. First, the accounting framework separates features of units' dynamics: each of its components converges to one term of the law of motion. Because drift weights units' changes by their initial weights, set before the shocks occur, the shocks average out of it: in the limit, drift is the first moment of units' changes integrated against the slope of the influence function.[8]

Second, displacement and demography are not artifacts of observing units at discrete dates. Like drift, they do not vanish per unit of time as the interval shrinks. Displacement is second order in units' changes, but squared changes are first order in time: with shocks, the variance of a unit's change grows in proportion to the interval, as for a random walk, so the average squared change over a month is about a twelfth of that over a year. Demography is first order in time because units enter and exit at a positive rate. Over short intervals, each component is therefore proportional to the length of the interval, up to terms of second order in time (Section 2.2).

Third, displacement takes a particularly simple form: it is the variance of units' shocks, weighted by half the curvature of the influence function (Table 2 gives it for the statistics of Table 1). The simple form also signs displacement: it is non-negative for any volatility of units' shocks if and only if the influence function is convex in position, $\partial_{xx}\mathrm{IF}_\nu(x; G_t) \geq 0$. The influence function is convex at every distribution if and only if the statistic satisfies the *transfer principle* (also called the Pigou–Dalton principle): a transfer from a higher to a lower position, without reversing their ranking, does not increase the statistic (Dalton, 1920). Shocks do the reverse, spreading out units that start at the same position without changing their mean, so they raise any statistic that satisfies this principle (Rothschild and Stiglitz, 1970; Atkinson, 1970).

[7] This is a population result: with a continuum of units, independent shocks average out. In a sample of N units with weights of order $1/N$, the components also have sampling error, which is separate from this limit.

[8] Weights set later would be correlated with the shocks they weight, so drift would also contain their second moment: with final weights, it would converge to drift plus twice displacement; with the average of initial and final weights, as in the trapezoid rule of Section 1.1, to drift plus displacement.

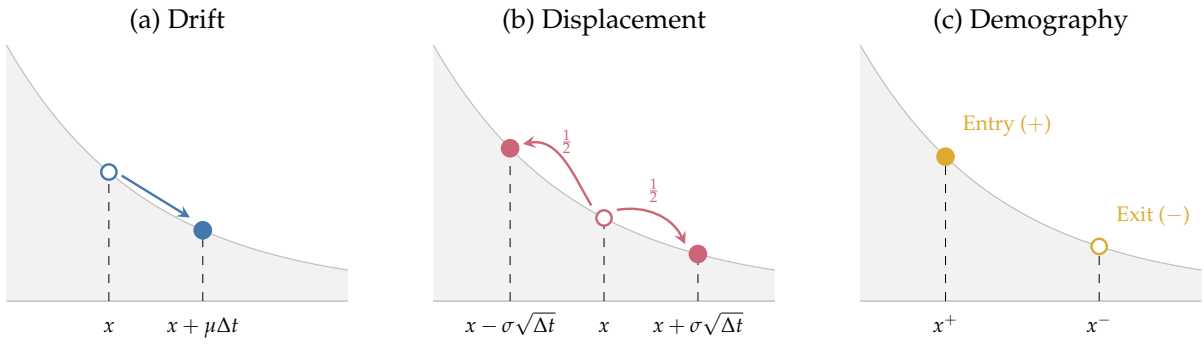


Figure 1: How Drift, Displacement, and Demography Move the Statistic

Notes: The curve is the density of positions; hollow and filled dots mark initial and final positions. Effects are per unit of mass, with $\mu = \mu_t(x)$, $\sigma = \sigma_t(x)$, and IF_ν evaluated at G_t. Panel (a) moves mass from x to $x + \mu\Delta t$, which changes the statistic by about $\partial_x \mathrm{IF}_\nu(x; G_t)\mu\Delta t$. Panel (b) splits mass equally between $x - \sigma\sqrt{\Delta t}$ and $x + \sigma\sqrt{\Delta t}$: the linear effects cancel, leaving a leading effect of $\frac{1}{2}\partial_{xx}\mathrm{IF}_\nu(x; G_t)\sigma^2\Delta t$. In panel (c), entry at x^+ adds $\mathrm{IF}_\nu(x^+; G_t)$ and exit at x^- subtracts $\mathrm{IF}_\nu(x^-; G_t)$. Integrating the effects of panels (a) and (b) over G_t and dividing by Δt gives drift and displacement; integrating those of panel (c) against the rates Ψ_t^+ and Ψ_t^- gives demography.

Table 2: Drift and Displacement for Common Statistics in Continuous Time

Cross-sectional statistic		Decomposition of $\mathrm{d}\nu(G_t)\,/\,\mathrm{d}t$	
Name	ν	Drift	Displacement
Average	$\mathrm{E}[x_{it}]$	$\mathrm{E}[\mu_t(x_{it})]$	0
Variance	$\mathrm{Var}(x_{it})$	$\mathrm{Cov}(2x_{it}, \mu_t(x_{it}))$	$\mathrm{E}[\sigma_t(x_{it})^2]$
Average product	$\mathrm{E}[x_{it}y_{it}]$	$\mathrm{E}[y_{it}\mu_{xt}] + \mathrm{E}[x_{it}\mu_{yt}]$	$\mathrm{E}[\rho_t\sigma_{xt}\sigma_{yt}]$
Average of a function	$\mathrm{E}[f(x_{it})]$	$\mathrm{E}[f'(x_{it})\mu_t(x_{it})]$	$\frac{1}{2}\mathrm{E}[f''(x_{it})\sigma_t(x_{it})^2]$
Exponential mean ($\omega_{it} \equiv e^{\gamma x_{it}}/\mathrm{E}[e^{\gamma x_{it}}]$)	$\frac{1}{\gamma}\ln \mathrm{E}[e^{\gamma x_{it}}]$	$\mathrm{E}[\omega_{it}\mu_t(x_{it})]$	$\frac{\gamma}{2}\mathrm{E}[\omega_{it}\sigma_t(x_{it})^2]$
Ordered weighted average	$\mathrm{E}[w(R_{it})x_{it}]$	$\mathrm{E}[w(R_{it})\mu_t(x_{it})]$	$\frac{1}{2}\mathrm{E}[w'(R_{it})g_t(x_{it})\sigma_t(x_{it})^2]$
Top p average	$\mathrm{E}[x_{it} \mid R_{it} \geq 1-p]$	$\mathrm{E}[\mu_t(x_{it}) \mid R_{it} \geq 1-p]$	$\frac{1}{2p}g_t(q_{1-p,t})\sigma_t(q_{1-p,t})^2$
Gini	$\mathrm{E}[(2R_{it}-1)x_{it}]$	$\mathrm{E}[(2R_{it}-1)\mu_t(x_{it})]$	$\mathrm{E}[g_t(x_{it})\sigma_t(x_{it})^2]$

Notes: Each row gives the drift and displacement terms of Proposition 2 for the statistic in the same row of Table 1, with positions following the diffusion of Assumption 1: expected change $\mu_t(x)$ and volatility $\sigma_t(x)$. g_t is the density of positions and $q_{1-p,t}$ the threshold of the top p, their $(1-p)$-quantile. In the average product, μ_{xt}, σ_{xt} and μ_{yt}, σ_{yt} are the expected changes and volatilities of the two variables, and ρ_t the correlation of their shocks (Appendix B.1). Appendix E decomposes more statistics and gives demography.

2.2 Beyond Diffusions

I now discuss the law of motion of the statistic beyond the case where units follow diffusions with idiosyncratic shocks. I extend Proposition 2 to idiosyncratic jumps and to common and granular shocks. Finally, I examine the gap between the accounting over an interval and its continuous-time limit, which is of second order in time.

Jumps. The baseline model assumes that paths are continuous, but positions can also jump: a firm's productivity when it innovates, a household's wealth with an inheritance or the sale of a business. A jump does not shrink with the interval, so displacement then depends on all moments of units' changes, not only their variance: the n-th derivative of the influence function weights the n-th moment of jumps (Proposition B1 in Appendix B.2).

Common and granular shocks. When many units share a shock, or a few large units carry much of the weight in a statistic, drift carries most of the statistic's fluctuations over short intervals. The baseline model rules out both cases: with idiosyncratic shocks and infinitesimal units, shocks average out of drift. Common and granular shocks do not.[9] Drift then includes the realized shocks, which are of the order of the square root of the interval, whereas displacement, which sums squared changes, and demography are of the order of the interval. Drift is therefore random: with granular shocks, its variance across intervals is the squared granular volatility of the statistic (Gabaix, 2011); averaging it over many intervals recovers the contribution of expected changes. Displacement and demography instead contribute steadily to the trend of the statistic.

Both kinds of shocks also add a term to displacement. A common shock moves units at different positions by different amounts: a rise in stock prices, for example, raises households' wealth in proportion to their equity holdings. Unlike the dispersion among units that start at the same position, its nonlinear effect has no fixed sign (Proposition B2 in Appendix B.3). A granular shock moves the totals that the statistic depends on, such as market sales, and with them the large unit's own weight; this term matters when a few units carry most of the weight (Appendices B.3, D.3, and E.5).

Accounting versus instantaneous rates. Integrating the rates of Proposition 2 over an interval gives a second exact decomposition of the same change. It agrees with the accounting of Section 1.1 to first order in time but differs from it at second order in time, for two reasons. First, the accounting holds each unit's weight at its initial value over the whole interval, whereas the integrated rates update it as the unit moves. Displacement over the interval therefore exceeds the integral of its rate when units whose weights have risen keep gaining, as with persistent differences in growth, and falls short of it under mean reversion. Second, the accounting counts the changes of units that enter or exit during the interval in demography,

[9] Drift sums units' shocks with their weights. The shocks average out when each unit's weight is small and shocks are independent, or correlated only within small groups of units. They do not when many units share a shock or when a few large units carry much of the weight (Gabaix, 2011).

whereas the rates count them in drift and displacement while these units are present; since exiters are selected on their changes, this gap does not average out (Appendix B.4). Because these gaps grow faster than the interval, the two decompositions are close over short intervals but can differ over long ones, so comparing a model with data requires computing the model's components over the same interval as the data.

3 Implementing the Decomposition with Micro Data

This section discusses the choices that arise when applying the decomposition to micro data: how to handle positions measured with noise or averaged over a period, how to define continuing units, and in which variable to write positions.

Accounting for measurement error. Positions in panel data are often measured with error. Measurement error biases the statistic itself, especially when the statistic is not linear in positions. More importantly here, it also biases the split between drift and displacement: drift pairs a unit's measured weight with its measured change, so an error in its initial position enters both. Consider classical error, with mean zero and independent across units, over time, and of true positions and changes. It creates spurious mean reversion: a unit whose position is overstated at t appears to fall at $t + \Delta t$. For statistics satisfying the transfer principle, this pushes drift down and displacement up (Proposition C1 in Appendix C.1).

Removing this bias requires a weight that does not contain the error at t, that is, a prediction of each unit's weight at t from its earlier positions. The simplest prediction is the weight one interval earlier. With these *lagged weights*, drift is

$$\text{Drift}^{\text{lag}} \equiv \sum_{i \in C} \frac{\partial \nu(G^C_{t-\Delta t})}{\partial x_{i,t-\Delta t}} \Delta x_{it}, \tag{3}$$

where C now denotes continuing units also observed at $t - \Delta t$, and displacement is the remainder, $\Delta\nu(G^C_t) - \text{Drift}^{\text{lag}}$. Since errors are independent over time, the weight at $t - \Delta t$ does not contain the error at t. The correction only moves terms between drift and displacement.

Lagged weights are less accurate when a unit's change predicts its next one, as when young firms grow faster than old firms of the same size: the correction then also moves a term of second order in time from drift to displacement. Appendix C.1 gives a correction that removes this term by extrapolating each unit's weight from its two past values.[10]

Another form of measurement error is that the data often record the average position over a period rather than its value at a date. This happens with flows, such as income or sales, as opposed to stocks, such as wealth or capital. Even without noise, flows bias the split the other way, by creating spurious momentum. If sales rise halfway through a year and then stay flat, next year's annual sales still exceed this year's. Recording period averages thus moves a third of one period's displacement into drift, whatever the interval between observations (Ap-

[10] When only some continuing units are observed at the earlier dates, either correction can be estimated on them, as a share of their displacement, and applied to all continuing units. The applications do so for lagged weights.

pendix C.1). In practice, however, this bias needs no separate correction: the lagged weights already used for noise remove it too, provided the lag covers at least one flow period, as it does in the applications.

Defining continuing units. The decomposition requires defining the set of continuing units. Often this is simple: they are the units observed at both dates. However, some events pass one unit's position to others: a death passes a household's wealth to its heirs, a merger a target's sales to its acquirer. If the receiving units continue, the event is split between several terms: exit records the deceased household or the target, while drift and displacement record the receiving units' jump. It is often more natural to count the receiving units as exiting at t and entering again at $t + \Delta t$, which records the whole event in demography. Demography can then be split by type of event, since to first order it is a sum over entrants and exiters (Section 2.1): firms' exit into closures and acquisitions; households' entry and exit into births and deaths, marriages and divorces, and moves into or out of the survey sample.

Choosing the variable. The same statistic can be written as a function of different variables: positions or positions relative to their mean, in levels or in logs. The choice leaves demography unchanged but moves terms between drift and displacement (Appendix C.2). Normalizing positions by the mean of continuing units moves only a term of second order in time. Another choice is to detrend positions, dividing each by the average across all units at that date. The average is then seen as a scale, outside the statistic, whose growth is taken as given. Drift then measures how units grow relative to this average. It therefore also includes the effect of entry and exit on the average, which is otherwise part of demography (Appendix E.6.2). Taking logs moves a Jensen term (an Itô term in continuous time), which is first order in time when units face shocks: drift then sums units' log growth rather than their growth, with the same weights. In each application, I use the variable whose weights are the familiar ones: normalized wealth and market shares in levels, whose weights are rank weights and shares; log productivity, whose weights are sales shares.[11]

4 Applications

This section applies the general accounting framework in three settings: aggregate productivity in U.S. firms, wealth inequality in Europe, and the concentration of U.S. consumer product markets. The statistics differ in form: industry productivity is an exponential mean, the Gini and top shares weight households by rank, and the HHI is a sum of squared shares. The same accounting framework relates the trend of each statistic to the micro data, even though its components have a different economic interpretation in each setting. Despite these differences, one finding is pervasive: in the productivity and wealth applications, displacement and

[11] The decomposition is additive. For a positive statistic, the same framework also decomposes its growth, $\nu(G_{t+\Delta t})/\nu(G_t)$, as the product of a drift, a displacement, and a demography factor. For statistics homogeneous of degree one, such as the top p average, growth common to all positions then moves only the drift factor (Appendix C.2).

demography are of the same order of magnitude as drift.

4.1 Aggregate Productivity

I start with aggregate productivity. In this setting, drift is firms' average improvement, their productivity growth weighted by their initial sales, which growth accounting computes with Hulten's formula; displacement is the growth from demand reallocating toward the firms whose productivity rose; demography is the growth from entering firms, net of those that exit. Firms' productivity is not observed, so measuring these components takes a model of firms and demand.

Productivity aggregator. Consider an industry composed of firms that make the same detailed product. Firms sell varieties that buyers substitute with a constant elasticity $\theta > 1$, produce with constant returns from the same bundle of inputs bought at the same prices, and charge the same markup over marginal cost. Let A_{it} denote firm i's effective productivity, its output per unit of the input bundle adjusted for the appeal of its variety. The industry's log productivity is then an exponential mean of its firms' log productivity (Table 1, with $\gamma = \theta - 1$),

$$\ln \bar{A}_t \equiv \frac{1}{\theta - 1} \ln \sum_i e^{(\theta-1)\ln A_{it}}.$$

Summing over firms rather than averaging adds the love-of-variety term $\ln N_t/(\theta-1)$, where N_t is the number of firms (Dixit and Stiglitz, 1977). It is the efficient productivity of Hsieh and Klenow (2009), which increases with the dispersion of productivity across firms (Appendix D.1.4). Proposition 1 gives the following decomposition.

Corollary 1 (Decomposition of Industry Productivity)**.** *The change in productivity over time in each industry can be decomposed as follows:*

$$\Delta \ln \bar{A}_t = \underbrace{\sum_{i\in C} \frac{s_{it}}{s_{Ct}} \Delta \ln A_{it}}_{\text{Drift (= Hulten)}} + \underbrace{\frac{1}{\theta-1} \sum_{i\in C} \frac{s_{it}}{s_{Ct}} \ln \frac{A_{it}^{\theta-1} / \sum_{j\in C} A_{jt}^{\theta-1}}{A_{i,t+\Delta t}^{\theta-1} / \sum_{j\in C} A_{j,t+\Delta t}^{\theta-1}}}_{\text{Displacement}} + \underbrace{\frac{1}{\theta-1} \ln \frac{\sum_{i\in C} A_{it}^{\theta-1} / \sum_i A_{it}^{\theta-1}}{\sum_{i\in C} A_{i,t+\Delta t}^{\theta-1} / \sum_i A_{i,t+\Delta t}^{\theta-1}}}_{\text{Demography}},$$

where C is the set of firms of the industry present at both dates, s_{it} is firm i's share of the sales of the industry, and $s_{Ct} \equiv \sum_{i\in C} s_{it}$ is the combined share of the firms of C.

Among continuing firms, the marginal effect of a firm's log productivity is $A_{it}^{\theta-1} / \sum_{j\in C} A_{jt}^{\theta-1}$, its weight. Firms charge the same markup over the same costs, so this weight is the firm's share of continuing firms' sales, s_{it}/s_{Ct}. Drift weights firms' log productivity growth by these initial shares: it is the usual Hulten (1978) term, firms' average improvement.[12] Displacement is the gain from demand reallocating toward the firms whose productivity rose. For small changes, it is approximately $(\theta-1)/2$ times the variance of log productivity changes, weighted by the

[12] Displacement and demography also depend on the final weights, which Corollary 1 writes with productivity: they equal final sales shares when productivity includes the appeal of varieties at each date, as in (4) below, but not when productivity is measured at fixed appeal (Appendix D.1.5).

initial weights.[13] It therefore has the sign of $\theta - 1$: dispersion in firms' productivity changes raises industry productivity when their varieties are substitutes, since the exponential mean then gives more weight to high log productivity, and would lower it if they were complements ($\theta < 1$). Demography is positive when entrants take a larger share of sales than exiters held, because they are more numerous or more productive. It is the correction for new and disappearing varieties of Feenstra (1994); Aghion et al. (2019) use a related expression to measure growth from creative destruction.

These components differ from the terms of the standard decomposition of productivity growth, by Foster et al. (2001), which splits the change in the share-weighted average of firms' productivity into within, between, and cross terms, plus entry and exit. Their within term is drift. Their cross term pairs changes in shares with changes in productivity, like displacement, but also counts as a gain the concentration of sales that dispersion creates, which industry productivity offsets by the loss of variety: when productivity is inferred from sales shares, as below, it is about twice displacement (Appendix E.7). The cross term is also often interpreted as noise, or as an interaction of two changes that vanishes over short intervals (Foster et al., 2001), whereas displacement is first order in time (Section 2.1) and lagged weights separate it from noise (Section 3).

Decomposition in existing growth models. Table 3 decomposes the instantaneous rate of aggregate productivity growth in a benchmark with identical technology (Panel A) and in nine published models of firm dynamics at their authors' calibration (Panels B and C), with product lines as units. A line continues whoever owns it, so an innovation by a new producer is a change in the line's quality, and demography records only lines that appear or disappear. In these models, aggregate productivity growth almost always differs from drift, the sales-weighted average of lines' productivity growth that Hulten's formula measures. Models that reach similar growth rates also split it in very different ways.

Drift is the growth that the same productivity changes would bring if each line kept its initial share of spending, as with Cobb–Douglas demand; displacement is the extra growth from substitution. Displacement is positive in most of these models; it rises with the elasticity θ and with the dispersion of lines' productivity changes.[14] If each line's log productivity jumps by a random step J at Poisson rate λ, independently of its size, drift is $\lambda \mathrm{E}[J]$ and displacement is $\frac{1}{2}(\theta-1)\lambda \mathrm{E}[J^2] + \frac{1}{6}(\theta-1)^2 \lambda \mathrm{E}[J^3] + \cdots$ (Appendix D.1.1). For the same average improvement, an economy whose firms pursue riskier R&D, with rare large steps rather than frequent small ones, therefore grows faster: a standard deviation of productivity growth of 5 percent a year adds about 0.4 percentage points of growth a year at $\theta = 4$.

[13] Outside CES, the three components still exist in an efficient economy, and drift still weights productivity changes by initial sales shares (Hulten, 1978); displacement and demography then depend on sales shares along the path from initial to final productivity, which CES pins down.

[14] The mechanism recalls the fundamental theorem of natural selection (Fisher, 1930): a population's mean fitness rises at a rate equal to the variance of fitness, as fitter types multiply. In Fisher's theorem, the variance is across persistent types, which here would show up in drift, as the covariance between weights and changes in (1); displacement instead arises even when all lines have the same expected growth, from the dispersion of outcomes among lines that start in the same place.

Table 3: Decomposing Productivity Growth in Growth Models

	Total	Drift	Displacement	Demography
Panel A: Identical technology				
Solow (1956)	2.00	2.00	0.00	0.00
Panel B: Quality-ladder models				
Aghion et al. (2019)	2.71	1.86	0.65	0.20
Garcia-Macia et al. (2019)	1.75	1.24	0.36	0.15
Peters (2020)	3.00	3.00	0.00	0.00
Peters and Walsh (2026)	1.84	3.16	0.11	−1.43
Panel C: Idea-diffusion models				
Luttmer (2007)	2.00	0.59	1.05	0.36
Lucas and Moll (2014)	2.03	0.87	1.16	0.00
Sampson (2016)	1.55	0.00	0.00	1.55
Perla et al. (2021)	0.79	−1.20	2.00	0.00
Chen (2026)	2.00	0.00	2.00	0.00

Notes: Instantaneous rates, in percent per year, on each model's balanced growth path at its authors' calibration, which Chen (2026) reaches only asymptotically. Total is the sum of the other three columns; rounded entries may not add up exactly. Appendix D.1.1 describes how each model is implemented.

Displacement rises from Panel A to Panel C. In Panel A, all lines share the same technology, so no line gains on another and displacement is zero. Quality-ladder models (Panel B) picture growth as frequent, incremental improvements to existing products. Steps are small, so displacement is a fraction of drift.[15] It is zero in Peters (2020), where the aggregate is linear across lines, as in quality-ladder models with Cobb–Douglas aggregation (Aghion and Howitt, 1992; Grossman and Helpman, 1991; Klette and Kortum, 2004). Models of idea diffusion (Panel C) picture growth as catching up: laggards learn from or adopt the technology of better firms, so a few firms make large gains at any time. Displacement then usually exceeds drift. Drift can even be negative. In Perla et al. (2021), firms' productivity decays until they adopt a better technology. Drift weights the jump at adoption by the firm's small share before it adopts, so Hulten's formula would show firms becoming less productive on average while aggregate productivity rises.

The table reports instantaneous rates: displacement and demography survive the continuous-time limit (Section 2.1; Appendix B.4). In Luttmer (2007), firms' log productivity follows a Brownian motion, whose paths are continuous, yet displacement exceeds drift: chaining Hulten's formula over ever shorter intervals would miss most of growth. Over one year, the interval of the data, the components are close to these rates (Table D1 in Appendix D.1.1).

I now turn to demography, which also differs widely across models. In Sampson (2016), incumbents' productivity is fixed for life, so all growth comes from demography, mostly through selection: lines that fall behind exit, and entrants, who draw from the incumbents' distribution, arrive far more productive. In Perla et al. (2021), the same catching up is displacement: a firm that falls behind copies a better technology and keeps its identity. Aggregate growth cannot

[15] With product lines as units, improvements by a line's owner and creative destruction, which Garcia-Macia et al. (2019) separate, are both quality steps on an existing line: displacement depends on how dispersed the steps are, not on who makes them (Appendix D.1.1).

tell the two apart, but a panel can.[16]

Empirical implementation. Measuring these components in firm data takes several assumptions. First, sales shares measure effective productivity, which includes the appeal of varieties: within a detailed product, firms buy inputs at the same prices and charge the same markup, so shares move only with productivity. Second, firms stand in for product lines: a line that changes hands is part of measured drift and displacement; what happens among a firm's lines is part of its own productivity growth.[17] Third, a 4-digit SIC code is a market of close substitutes with a common elasticity. Fourth, listed firms represent all firms. Under these assumptions, a firm's sales are proportional to $A_{it}^{\theta-1}$ within its industry, so the final weights in Corollary 1 are also sales shares:

$$\Delta \ln \bar{A}_t = \underbrace{\sum_{i \in C} \frac{s_{it}}{s_{Ct}} \Delta \ln A_{it}}_{\text{Drift (= Hulten)}} + \underbrace{\frac{1}{\theta - 1} \sum_{i \in C} \frac{s_{it}}{s_{Ct}} \ln \frac{s_{it}/s_{Ct}}{s_{i,t+\Delta t}/s_{C,t+\Delta t}}}_{\text{Displacement}} + \underbrace{\frac{1}{\theta - 1} \ln \frac{s_{Ct}}{s_{C,t+\Delta t}}}_{\text{Demography}}. \tag{4}$$

Sales shares therefore identify displacement and demography without any estimate of firms' productivity. Displacement is the relative entropy between continuing firms' initial and final shares, divided by $\theta - 1$: it measures how much these shares shift.[18] Demography likewise depends only on continuing firms' combined share.

I use listed firms in Compustat incorporated in the United States, from five years after their listing, and define a detailed product as a 4-digit SIC code with at least two listed firms. Mergers and restatements can move sales shares without a change in productivity, so I exclude merger targets and acquirers around consolidation and screen discrepancies between annual and quarterly sales (Appendix D.1.2). I set $\theta = 4$, the median elasticity within product categories in Hottman et al. (2016) and the value of Aghion et al. (2019). The elasticity enters the data differently from the models of Table 3: for a given dispersion of productivity changes, displacement rises with θ, but for given changes in sales shares it falls with θ, since the same changes in shares then imply smaller productivity changes (Appendix D.1.5).

Transitory noise can inflate relative entropy, while annual sales, which are flows, smooth within-year movements and can understate it. I correct both effects with the lagged weights of Section 3, evaluating each firm's share one interval before the start. The correction raises displacement by about half (Panel B of Table D2); quarterly sales confirm its size (Appendix D.1.2).

The data can test whether firms' sales move along smooth paths or with shocks over the horizons at which productivity is measured: along smooth paths, displacement over an interval shrinks with its square, whereas with shocks it is proportional to the interval (Section 2.1).

[16] Selection is negative in Peters and Walsh (2026), where entrants start well below the average line, enough to make demography negative: drift exceeds growth. In Romer (1990), new lines are like existing ones, so demography is pure mass, the gain from love of variety.

[17] The decomposition remains exact with firms as units, but its split differs (Appendix D.1.5).

[18] Displacement can also be written $K(\theta-1)/(\theta-1) - K'(0)$, where $K(u) \equiv \ln \sum_{i \in C} \frac{s_{it}}{s_{Ct}} e^{u \Delta \ln A_{it}}$ is the cumulant generating function of continuing firms' log productivity changes, weighted by their initial sales shares: it collects their cumulants of order two and higher (Appendix E.5). The higher ones matter little: a variance approximation exceeds relative entropy by only 7 percent (Appendix D.1.2).

Compustat reports firms' quarterly sales, so displacement can be computed over horizons of one to eight quarters on the same firms. Annualized displacement is close to constant across horizons, so displacement is close to linear in the horizon rather than quadratic (Figure 2).

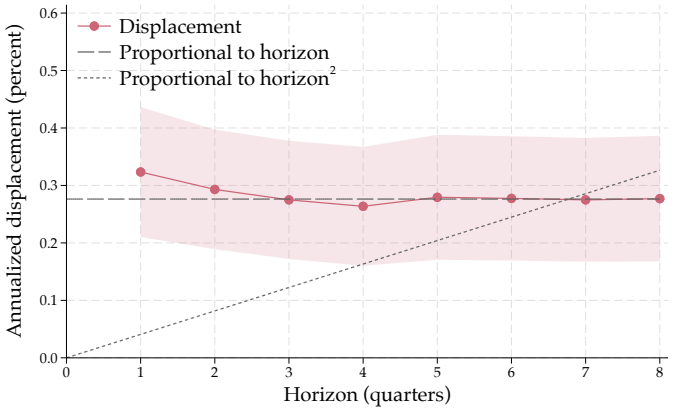


Figure 2: Displacement by Horizon

Notes: Displacement from quarterly Compustat sales, in percent per year, annualized by $4/H$ for a horizon of H quarters, with $\theta = 4$ and Domar weights. Each horizon uses the same firms, seasonally adjusted sales, and weights four quarters before the start. Long-dashed line: the annualized fit of displacement to aH, as with diffusions or jumps. Short-dashed line: the annualized fit to a multiple of H^2, as with paths that are smooth over these horizons. A quadratic term accounts for at most about a tenth of displacement over eight quarters. Shaded area: 95 percent confidence interval from a bootstrap over industries. Sample and fits: Appendix D.1.2.

Firms' productivity is not observed, so I measure displacement, demography, and total growth separately and obtain the Hulten term as the rest. Displacement is (4) within detailed products, aggregated across them with Domar weights, taking demand across the detailed products of a BEA industry to be Cobb–Douglas. Demography comes from the Census Business Dynamics Statistics, which cover all U.S. firms, with firms entering at age five, since a new firm's sales understate its productivity until it builds its customers (Aghion et al., 2019). Total growth is the Solow residual (U.S. Bureau of Economic Analysis, 2026; Fernald, 2014) plus the gains from substitution and variety that deflators omit: a fixed-basket price index within a detailed product misses buyers' substitution across firms and the arrival of new varieties. I infer these gains from sales shares under the benchmark of an arithmetic Laspeyres index within detailed products (Appendices D.1.3 and D.1.5). The Hulten term, as the rest, also contains displacement across BEA industries, which is close to zero over a year (Appendix D.1.4).

Table 4 reports the decomposition of aggregate productivity growth. Under these assumptions, most aggregate productivity growth comes from displacement and demography rather than from firms' average improvement: demand reallocating toward the firms whose productivity rose and the entry of firms, net of those that exit, account for about 70 percent of growth, which averages 1.18 percent a year from 1988 to 2023. Displacement contributes 0.37 percent a year, demography 0.48, and the residual Hulten term 0.33. As Section 2.2 predicts, displacement and demography are steadier than the residual Hulten term: they contribute substantially to average growth but little to its annual fluctuations (Figure 3).

I then use the decomposition to study the productivity slowdown (Fernald, 2015). Comparing the first period of Table 4, 1988–1999, with the other two, 2000–2023, aggregate productivity growth falls by 0.61 percentage points a year. Lower displacement accounts for about

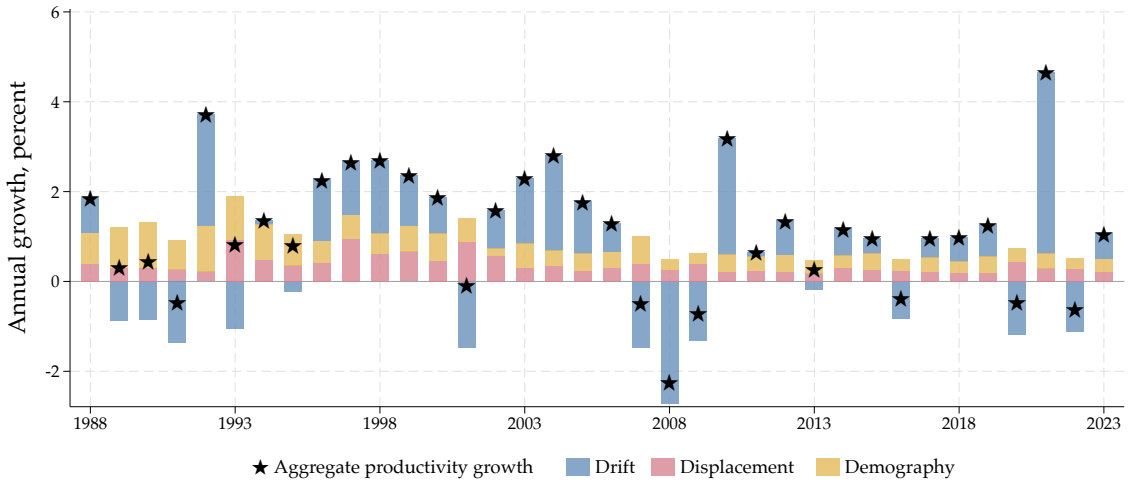


Figure 3: Decomposing Aggregate Productivity Growth by Year

Notes: Annual growth in percent; year t is the change from $t-1$ to t. Stars: aggregate productivity growth, the measured Solow residual plus the price bias of (D1) and demography, summed with final-expenditure shares under the deflator benchmark. Displacement: (4) within detailed products, from Compustat sales shares, with $\theta = 4$ and Domar weights, corrected for flows and noise with lagged weights (Section 4.1); demography: all firms, from the Census Business Dynamics Statistics; drift: the rest (Appendix D.1.3). Positive components are stacked above zero and negative ones below. Standard deviations across years: displacement 0.20 and demography 0.24 percentage points, against 1.40 for total growth and 1.38 for drift.

Table 4: Decomposing Aggregate Productivity Growth

	Total	Drift	Displacement	Demography
1988–1999	1.58	0.36	0.49	0.73
2000–2011	1.00	0.23	0.38	0.39
2012–2023	0.94	0.39	0.25	0.30
All years	1.18	0.33	0.37	0.48

Notes: Annual growth in percent, averaged over each period; year t is the change from $t-1$ to t. Components as in Figure 3, with the elasticity of substitution between firms within a detailed product $\theta = 4$; the total uses the deflator benchmark and is the sum of the last three columns.

30 percent of this decline and lower demography for about 60 percent; the residual Hulten term accounts for the rest.[19] In the model, the fall in displacement reflects less dispersion in firms' productivity growth and hence smaller gains from demand reallocating toward the firms whose productivity rose. Displacement measured from sales shares would also fall if sales responded less to the same productivity changes: Decker et al. (2020) find that businesses' employment has become less responsive to productivity shocks, whose dispersion did not fall. Hulten's formula, taken at face value, would attribute the slowdown to slower improvement of firms; the decomposition points instead to this lower dispersion and to less entry.

The appendix examines how the results depend on the four assumptions behind (4). With heterogeneous markups, alternative market definitions, or domestic sales only, displacement and demography are similar (Appendices D.1.5 and D.1.5). If changes in appeal are no gain to buyers, displacement also contains changes in tastes (Appendix D.1.5). If sales growth is at least as dispersed among unlisted firms as among listed ones, the listed-firm estimate under-

[19] If the sample of firms is held fixed in each comparison of consecutive years and these comparisons are chained, so that firms entering or leaving the sample cannot drive the decline, lower displacement accounts for about 40 percent of the decline.

states displacement (Appendix D.1.5). The one assumption the data cannot relax is that firms stand in for product lines, since Compustat has no sales by product line. Its direction is unclear: lines changing hands raise measured displacement, while changes among a firm's lines do not appear in it (Appendix D.1.5).

Displacement within versus across markets. The decomposition applies at every level at which units are aggregated: product lines within firms, firms within detailed products, detailed products within industries, and industries within the economy. At each level, displacement is, for small changes, $(\theta - 1)/2$ times the variance of units' log productivity changes, with θ the elasticity of substitution among the units of that level. Among continuing units, aggregate productivity growth is therefore the average improvement of the finest units plus the sum of these terms over levels. Average improvement is also relative to the unit: the Hulten term of firms contains the displacement among their product lines (Appendix D.1.5); likewise, Hulten's formula applied to industries counts the displacement and demography of their firms as the industries' own improvement.

Displacement and Hulten's theorem. The sign of displacement at each level of aggregation depends on whether its units are substitutes or complements. Baqaee and Farhi (2019) show that Hulten's theorem is a first-order approximation in productivity changes, with second-order effects whose size and sign depend on elasticities of substitution; in the CES industry of Corollary 1, displacement is the exact nonlinear remainder, whose leading term is their second-order correction. In their applications, this correction is negative and large for big shocks, such as the oil shocks of the 1970s, and for long-run growth across sectors, where complementarity lets slow-growing sectors gain weight: with weights fixed at their 1948 values, aggregate TFP would have grown by about 19 percentage points more by 2014 (the cost disease of Baumol, 1967; Baqaee and Farhi, 2019, Section 6.3). Within industries, where firms are substitutes ($\theta > 1$), the models of Table 3 and the data of Table 4 show the opposite sign: the nonlinearity is a source of growth, not only of fragility.

Levels of aggregation also differ in how displacement grows with the horizon: being second order in productivity changes does not make it second order in time. Within detailed products, displacement is proportional to the horizon from one quarter to two years, as with shocks (Figure 2). Across industries, it is close to zero over a year and becomes negative over longer horizons, because differences in industries' growth persist for several years (Figure D1 in Appendix D.1.4). Chaining Hulten's formula annually therefore captures nearly all of the displacement across industries but none of the displacement within detailed products. The continuous-time chaining of Hulten's formula in Baqaee and Farhi (2019, footnote 45) likewise misses displacement when firms receive shocks at every date, however small, as in the models of Table 3 and in Compustat sales.[20]

[20]Chaining with the average of initial and final weights, as a Törnqvist index does, would instead converge to drift plus displacement (Section 2.1). This chained sum no longer measures firms' average improvement, since its weights respond to the shocks they weight. Statistical agencies do not compute such an index across firms within a detailed product; one computed from prices would also leave out the displacement that comes from the dispersion

4.2 Wealth Inequality

I next use the accounting framework to study wealth inequality, measured by the Gini coefficient and the top 10% share. In this setting, drift measures how existing inequality evolves, whether initially rich households grow faster than poorer ones; displacement, the creation of inequality among households with the same initial wealth; demography, households that form or dissolve. I apply the decomposition to European household panels, which cover the whole wealth distribution and oversample wealthy households. The panel separates these forces, which a change in inequality alone does not.

Decomposing the Gini and top shares. Both are ordered weighted averages: they weight households' shares of wealth according to their percentile ranks (Weymark, 1981; Yaari, 1987). With s_{it} household i's share of wealth and R_{it} its percentile rank, the statistic is

$$\nu(G_t) \equiv \sum_i w(R_{it}) s_{it},$$

with $w(R) = \mathbf{1}_{R \geq 1-p}$ for the top p share and $w(R) = 2R - 1$ for the Gini. Each household's weight is its rank weight. Since the statistic sums wealth shares times these weights, its change among continuing households splits exactly into the change in shares at the initial weights, which is drift, and the change in the weights at the final shares, which is displacement. Applying Proposition 1 gives the following decomposition.

Corollary 2 (Decomposition of Ordered Weighted Averages)**.** *The change in an ordered weighted average over time can be decomposed as follows:*

$$\Delta\nu(G_t) = \underbrace{\sum_{i\in C} w(R^C_{it})\Delta\frac{s_{it}}{s_{Ct}}}_{\text{Drift}} + \underbrace{\sum_{i\in C}\frac{s_{i,t+\Delta t}}{s_{C,t+\Delta t}}\Delta w(R^C_{it})}_{\text{Displacement}} + \underbrace{\Delta\Big(\frac{1}{s_{Ct}}\Big(\sum_{i\notin C} s_{it}(w(R_{it}) - \nu(G_t)) + \sum_{i\in C} s_{it}(w(R_{it}) - w(R^C_{it}))\Big)\Big)}_{\text{Demography}},$$

where C is the set of households present at both dates, s_{it} and R_{it} are household i's share of wealth and percentile rank among all households, R^C_{it} is its percentile rank among the households of C, $s_{Ct} \equiv \sum_{i\in C} s_{it}$ is the combined share of the households of C, and the sums over $i \notin C$ run over the households that exit, at t, and over those that enter, at $t + \Delta t$. The term in parentheses is the gap between the statistic among all households and among those of C, so demography is entry plus exit (Section 1.1).

Drift holds each household's rank at its initial value. For the top p share, it is the change in the share of continuing households' wealth held by the households initially in the top p. For the Gini, it weights the change in each household's share of continuing households' wealth by twice its initial rank minus one. Relative gains that favor initially richer households raise drift; gains that favor poorer households lower it. Drift thus measures how existing inequality

of changes in appeal, which prices do not record (Appendix D.1.5).

evolves, whether initial gaps widen or close: it is the contribution of directional mobility, the systematic gains of one group relative to another (Jäntti and Jenkins, 2015).

Displacement is the creation of inequality among households with the same initial wealth: shocks open gaps among them, which, for statistics that weight households by rank, shows up as re-ranking. For the top p share, it is the final top group's share of continuing households' wealth minus the final share of the initial top group. It records the gains from replacing the households that leave the top with those that enter it.[21] For the Gini, displacement weights each change in rank by twice the household's final share of continuing households' wealth, so rank changes throughout the distribution count.[22] This is the contribution of exchange mobility, households overtaking one another. Displacement is non-negative for both statistics: assigning final ranks gives the largest weights to the largest final wealth holdings. It is zero when ranks do not change, even when households' changes differ. Shocks to returns and earnings can therefore sustain inequality even when initially poorer households grow faster, as in random growth models of wealth (Benhabib et al., 2011; Gabaix et al., 2016).

Demography is the effect of removing households that dissolve or leave and adding those that form. Entry and exit change both continuing households' shares of total wealth and their ranks in the population. Both statistics are population invariant, so entry and exit that replicate the population leave them unchanged: demography comes only from entrants and exiters that differ from continuing households. To first order, entry raises the statistic when entrants are concentrated in the tails of the distribution, and exit when exiters are concentrated in its middle (Appendix E).

Data and method. I use the Household Finance and Consumption Survey (HFCS; Household Finance and Consumption Network, 2026) for Germany, France, Italy, and Spain, the four largest euro-area economies and the survey's largest panels. The panels run to 2023 from around 2010 in Germany, Italy, and Spain and from around 2014 in France. For the UK, I use the Wealth and Assets Survey (WAS; Office for National Statistics, Social Survey Division, 2026) from about 2010 to 2022.[23]

Wealth is net wealth, excluding UK pension wealth for comparability with the HFCS. I decompose the change between consecutive waves, with wealth normalized by its mean in each wave and households observed in both waves reweighted to represent all continuing households. I annualize the components and average them over wave pairs within countries, then across countries with equal weights (Appendix D.2.1). Demography is the rest of the cross-sectional change, so it includes changes in survey composition as well as household turnover: entry also absorbs the sizable gap between re-interviewed households and the later

[21] I apply this decomposition to the Forbes 400 in Gomez (2023); Appendix E.6.2 compares the treatment of entry and exit.

[22] Each Gini component averages the corresponding components of top shares (Appendix E).

[23] This paper uses data from the Eurosystem Household Finance and Consumption Survey. The results published and the related observations and analysis may not correspond to results or analysis of the data producers. It also uses the Wealth and Assets Survey of the Office for National Statistics, distributed by the UK Data Service (SN 7215); the original data creators, depositors or copyright holders, the funders of the data collections, and the UK Data Archive bear no responsibility for their further analysis or interpretation.

cross-section, which cannot be separated from true entry (Appendix D.2.1).

Measurement error makes households whose wealth is understated look poor initially and appear to grow faster afterward. This spurious catching up lowers drift and raises displacement, so I evaluate the rank weights at wealth in the previous wave, as in Section 3 (Appendix D.2.1).

Results. Figure 4 reports the decomposition in each country, and Table 5 its average across countries, with standard errors. Both statistics move little on average across countries because displacement and demography offset negative drift: existing gaps close, but shocks open new ones and new households start with little wealth.

Table 5: Decomposing the Change in Wealth Inequality, Average across Countries

Gini				Top 10% share			
Total	Drift	Displ.	Demog.	**Total**	Drift	Displ.	Demog.
0.1	−1.1	0.6	0.6	0.1	−0.8	0.4	0.5
(0.1)	(0.2)	(0.2)	(0.1)	(0.1)	(0.2)	(0.2)	(0.1)

Notes: Displ.: displacement; Demog.: demography. Annual averages over wave pairs. Gini of normalized wealth, in points per year; share of wealth held by the richest 10 percent of households, in percentage points per year. Drift is corrected for measurement error with lagged weights. Equally weighted average of France, Germany, Italy, and Spain (HFCS) and the UK (WAS); Figure 4 shows each country. Standard errors in parentheses, from 200 bootstrap draws that redo every step, including the correction for measurement error: HFCS replicate weights, combined across implicates by Rubin's rules, and for the UK, Poisson weights drawn per household and kept across waves.

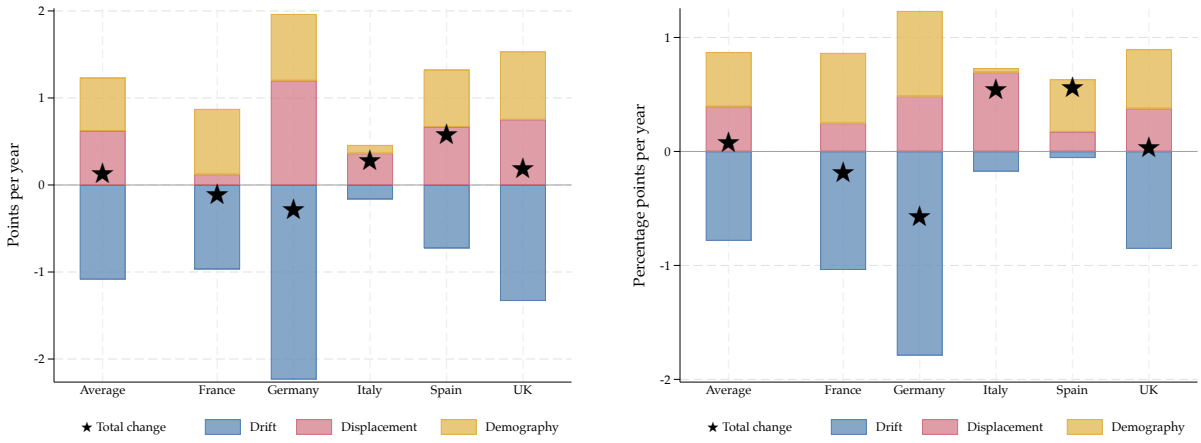
Drift averages −1.1 Gini points a year and −0.8 percentage points of the top 10% share across countries. The point estimate is negative in every country for both statistics (Figure 4). The interpretation is that initially poorer households gain on richer ones. Mean growth declines with wealth: households at the 25th percentile grow 3.3 percentage points a year faster than mean wealth, while those at the 99th percentile grow 2.1 points slower (Figure 5, panel (a)). The reason is that labor income is large relative to wealth at the bottom, so the same saving raises a poorer household's wealth by proportionally more (Fagereng et al., 2019). Saving also accounts for the pattern higher up the distribution: in Swedish registers, the relative decline of rich households below the top 0.1 percent comes from active saving (Bach et al., 2018).

Displacement averages 0.6 Gini points a year and 0.4 percentage points of the top 10% share. Idiosyncratic shocks in returns and earnings open gaps among households that start with similar wealth. Growth stays volatile throughout the distribution after removing measurement noise: its standard deviation is 32 percent a year at the 30th percentile and 14 to 16 percent between the 70th and 95th percentiles (Figure 5, panel (b)).

Demography averages 0.6 Gini points a year and 0.5 percentage points of the top 10% share. It comes from entry, which adds 0.6 Gini points a year, while exit, which records only identified exits, adds 0.0; the top 10% share splits in the same way. The interpretation is that new households start with little wealth, so their entry puts mass at the bottom of the distribution, which raises both statistics. In the HFCS countries, households under 35 that were not interviewed in the previous wave are 7 percent of households but hold 2 percent of wealth, with 83 percent of them in the lower half of the distribution. Not all of them are new: as discussed above, entry

also contains changes in survey composition.

The components separate two questions that are easily confused: whether poorer households catch up and whether inequality falls. Inferring falling inequality from negative drift alone is Galton's fallacy (Galton, 1886), which growth empirics knows as the gap between β- and σ-convergence (Quah, 1993): regression toward the mean need not reduce dispersion. The converse mistake is to infer from a stable statistic that poorer households do not catch up. Drift is what measures of mobility compute, as the instantaneous mobility measure of Genicot and Ray (2023) does. The other two components answer an additional question: how this mobility relates to the change in the statistic between two cross-sections. For their relative index, the statistic barely moves on average across countries, while its drift shows poorer households gaining (Appendix D.2.3). Two economies can also have the same stable Gini, one with every component zero and one with large offsetting components: in the second, households' positions are temporary, so wealth averaged over a decade is less unequal than wealth at a date (Friedman, 1962; Shorrocks, 1978).



(a) Gini

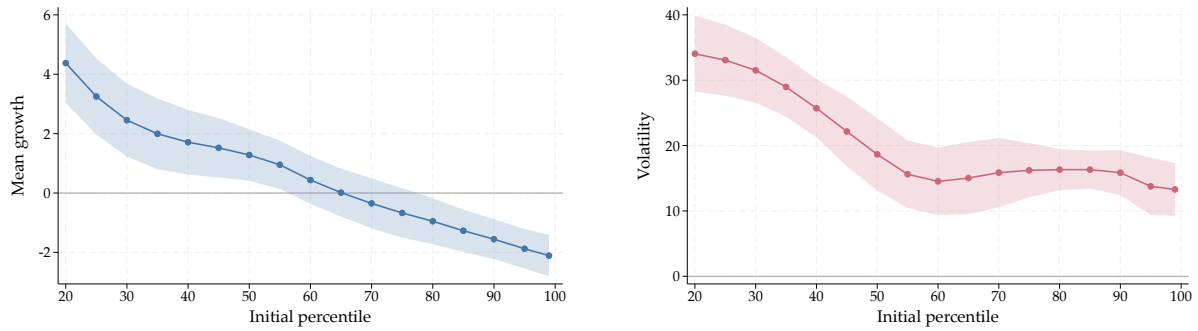
(b) Top 10% share

Figure 4: Decomposing the Change in Wealth Inequality by Country

Notes: Annual averages over wave pairs, as in Table 5, which reports their average across countries with standard errors. Components are stacked above zero when positive and below zero when negative; stars mark the total change. Panel (a): Gini of normalized wealth, in points per year. Panel (b): share of wealth held by the richest 10 percent of households, in percentage points per year. Drift is corrected for measurement error (Appendix D.2.1).

Decomposing existing models of the wealth distribution. In a stationary distribution, the components sum to zero but need not be zero, which gives a model of the wealth distribution two targets beyond the distribution itself and separates models that match the same distribution through different dynamics. Multiplying households' expected changes, the variance of their shocks, and the rates of entry and exit by a common factor leaves a stationary distribution unchanged but scales every component by that factor, so the components also pin down the speed of these dynamics. Since many such models already exist, I compute the components in published models rather than calibrate a new one.

Table 6 decomposes the Gini and the top 10% share in the stationary distributions of eight published models, measured as in the panels: over three years, with households that die



(a) Mean growth

(b) Volatility

Figure 5: Wealth Growth and Volatility by Percentile

Notes: Households observed in three consecutive waves (Appendix D.2.2). Growth is the annual arc-percent change of normalized wealth: the change divided by the average of its absolute values at the two dates, which approximates relative growth and remains defined for little or negative wealth. Kernel-weighted means at each percentile (triangular kernel, bandwidth 20 percentiles), averaged within each country, then with equal weights across the five countries. Panel (a): mean growth, by percentile of wealth in the previous wave. Panel (b): standard deviation of growth, net of measurement noise. Percent per year. Shaded areas: 95 percent confidence intervals.

counted as exiters, and with the same lagged weights (Appendix D.2.4). The three components sum to zero in every model, but their split differs widely across models. Drift is negative in most models, because young households save up from little wealth while older or richer ones spend down. Displacement comes mostly from shocks to returns, such as entrepreneurs' persistent high returns or investors' risky ones, which spread out households that start with the same wealth. Demography depends on how wealth passes across generations. It is positive when newborns start with no wealth, since their entry raises inequality. It is near zero when heirs inherit their parents' estates, so displacement alone offsets drift. In Blanchard (1985), the benchmark with finite lives alone, wealth depends only on age, so ranks never change and displacement is zero at any interval: demography alone offsets drift. The larger the offsetting components, the more the cross-section overstates the inequality of wealth averaged over time; this model is the pure case, in which all inequality is age.

Figure D2 in Appendix D.2.4 plots the Gini's drift and displacement in each model against the data. In the panels, displacement and demography both offset negative drift, as in models where newborns start with no wealth and returns are risky. Moll et al. (2022) comes closest. Three models miss the panels' negative drift: in Benhabib et al. (2011), Gabaix et al. (2016), and Peter (2026), drift is near zero or positive. Demography in the panels also contains changes in survey composition, so drift and displacement are the cleaner targets.

Table 6: Decomposing Wealth Inequality in Models of the Wealth Distribution

	Gini				Top 10% share			
	Total	Drift	Displ.	Demog.	**Total**	Drift	Displ.	Demog.
Blanchard (1985)	0.0	−1.0	0.0	1.0	0.0	−0.5	0.0	0.5
Cagetti et al. (2006)	0.0	−0.8	0.8	0.0	0.0	−0.8	0.8	0.0
Benhabib et al. (2011)	0.0	−0.1	0.2	−0.1	0.0	0.0	0.1	−0.1
Gabaix et al. (2016)	0.0	0.0	0.5	−0.5	0.0	0.0	0.2	−0.2
Moll et al. (2022)	0.0	−1.7	0.6	1.2	0.0	−1.0	0.3	0.7
Guvenen et al. (2023)	0.0	−1.3	1.3	0.0	0.0	−0.5	0.5	0.0
Gomez et al. (2024)	0.0	−3.4	0.3	3.0	0.0	−2.9	1.3	1.5
Peter (2026)	0.0	0.2	0.1	−0.2	0.0	0.2	0.1	−0.3
Data	0.1	−1.1	0.6	0.6	0.1	−0.8	0.4	0.5

Notes: Displ.: displacement; Demog.: demography. Each model at its published calibration, in its stationary distribution, measured as in Table 5: changes over three years, annualized, with households that die counted as exiters and drift corrected with lagged weights. The three components sum to zero; rounded entries may not add up exactly. Data: the cross-country average of Table 5. Appendix D.2.4 gives details.

4.3 Product Market Concentration

I finally turn to product market concentration, measured by the Herfindahl–Hirschman index (HHI). Merger guidelines use the HHI to screen for market power (U.S. Department of Justice and Federal Trade Commission, 2023). It is also motivated by theory: under Cournot competition, the HHI equals the elasticity of market demand times the sales-weighted average Lerner index, the markup over marginal cost as a share of price (Cowling and Waterson, 1976). In this setting, drift is leading brands gaining share; displacement, market-share instability, which raises concentration even when no brand is expected to outgrow the others; demography, the entry and exit of brands. All three can be computed from sales shares, without a model of firms or demand.

Decomposing the HHI. In a market, the HHI is the sum of the brands' squared shares of sales,

$$\mathrm{HHI}_t \equiv \sum_i s_{it}^2.$$

Applying the accounting of Section 1.1 to the HHI gives its decomposition in sales shares.

Corollary 3 (Decomposition of the HHI)**.** *The change in the HHI over time in each market can be decomposed as follows:*

$$\Delta\mathrm{HHI}_t = \underbrace{2\sum_{i\in C}\frac{s_{it}}{s_{Ct}}\Delta\left(\frac{s_{it}}{s_{Ct}}\right)}_{\text{Drift}} + \underbrace{\sum_{i\in C}\left(\Delta\frac{s_{it}}{s_{Ct}}\right)^2}_{\text{Displacement}} + \underbrace{\Delta\left(\sum_{i\notin C}s_{it}^2 - \left(\frac{1}{s_{Ct}^2}-1\right)\sum_{i\in C}s_{it}^2\right)}_{\text{Demography}},$$

where C is the set of brands of the market present at both dates, s_{it} is brand i's share of the sales of the market, $s_{Ct} \equiv \sum_{i\in C} s_{it}$ is the combined share of the brands of C, and the sums over $i \notin C$ run over the brands that exit, at t, and over those that enter, at $t+\Delta t$. The term in parentheses is the gap between the HHI of all brands and that of the brands of C, so demography is entry plus exit (Section 1.1).

Drift is positive when larger brands gain share. Several explanations of rising concentration predict this pattern: sales move toward the most productive firms (Autor et al., 2020), low interest rates make leaders invest more than followers (Liu et al., 2022), or slower diffusion of knowledge shields leaders from being copied (Akcigit and Ates, 2023).

The HHI is quadratic in shares, so displacement is exactly the sum of the squared changes in continuing brands' shares at any horizon. It is a measure of market-share instability, like the index of Hymer and Pashigian (1962), which sums the absolute changes in shares. Displacement measures the dispersion of sales growth among continuing brands, weighted by squared shares rather than by shares as for industry productivity, so large brands count for more. It raises concentration even when no brand is expected to gain share.

Demography is entry plus exit. Small entrants lower the HHI by diluting continuing brands' shares; exit reverses this effect (Appendix D.3). Declining entry can therefore raise concentration, whether entry falls because the labor force grows more slowly (Hopenhayn et al., 2022) or because barriers to entry rise (Gutiérrez and Philippon, 2019; Covarrubias et al., 2020).

The HHI can thus rise because leading brands gain share, because market shares become more unstable, or because entry weakens. These changes can have different implications for competition: concentration from barriers to entry signals market power, whereas concentration from the expansion of efficient firms or from more elastic demand can come with stable or rising competition (Covarrubias et al., 2020). Market-share instability is the starkest case: industrial organization has long viewed the turnover of market shares as a sign of competition (Hymer and Pashigian, 1962), yet instability raises the HHI, so a screen based on the HHI counts churn among brands as concentration. Neither the level nor the change in the HHI distinguishes these contributions. A falling HHI can, for example, combine leading brands gaining share with entry that dilutes them.

Data and results. The data and results of this application use the NielsenIQ Retail Scanner data and are omitted from this version pending NielsenIQ review.

5 Conclusion

Aggregate statistics such as productivity, wealth inequality, and market concentration all summarize a distribution of units, so their trends come from how these units move, enter, and exit. This paper proposes a framework that makes this link exact: the change in any distributional statistic decomposes into drift, displacement, and demography. Each term has a distinct micro origin: drift comes from units' expected changes, displacement from the dispersion of changes among units that start at the same position, and demography from entry and exit.

The first lesson is conceptual. We tend to credit the trend in a statistic to drift alone: aggregate productivity growth to firms' average improvement, rising inequality to the rich getting richer, rising concentration to larger brands gaining share. Yet part of productivity growth also comes from demand reallocating toward the firms whose productivity rose; part of inequality,

from gaps that shocks open among households that start with the same wealth; part of concentration, from market-share instability, brands winning and losing share. I show, in theory and in the data, that all three components are first order in time, so displacement and demography, not just drift, contribute to the long-run trend.

The second lesson is practical: I develop a simple method to measure each component using panel data, with a correction for measurement error. These measurements are useful in two ways. First, they help explain why a trend appears: competing explanations often predict different components, so comparing components across periods, markets, or countries can tell the explanations apart. Second, they help discipline economic models: two models can match the same statistic in steady state with very different drift, displacement, and demography. Matching the decomposition in the data helps ensure that a model matches the statistic for the right reason.

I presented the framework in a general way, in the hope that it proves useful beyond the three applications of this paper. To make this easy, the paper comes with a Stata command, `ddd`, that computes the decomposition using panel data for any statistic.

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Appendix

A Proof of Proposition 2

Proposition 2 has two parts: the law of motion of the statistic, and the convergence of the accounting components of Section 1.1 to its terms. The first part is an instance of the chain rule for functionals of a flow of measures, the "Itô formula along a flow of measures" of the mean-field game literature (Carmona and Delarue, 2018, Section 5.6 and Theorem 5.99), in which $\partial_x \mathrm{IF}_\nu$ plays the role of the Lions derivative. That formula is stated for probability measures, whose mass is fixed, and for functionals whose influence function is twice continuously differentiable. Here, entry and exit change mass; the influence function can also have kinks (top shares) or jumps (quantiles). The proof therefore takes two routes: it expands the influence function when that function is smooth, and the evolving density when it has kinks or jumps. The second route requires a smooth density.

The proof treats the two parts in turn: the first under Assumptions A1 and A2, the second under Assumption A3 in addition. For $\Delta t > 0$, let G_t^- denote the measure of the date-t positions of the units that exit in $(t, t+\Delta t]$, and $G_{t+\Delta t}^+$ the measure of the date-$(t+\Delta t)$ positions of the units that enter in that interval, so that $G_t = G_t^C + G_t^-$ and $G_{t+\Delta t} = G_{t+\Delta t}^C + G_{t+\Delta t}^+$.

Assumption A1 (Regularity)**.** The following three conditions hold.

(i) *Differentiability along the flow*: for any two measures H and H' among G_t, G_t^C, $G_{t+\Delta t}^C$, and $G_{t+\Delta t}$, $\nu(H') - \nu(H) = \int \mathrm{IF}_\nu(x; G_t)\, \mathrm{d}(H' - H)(x) + o(\Delta t)$.

(ii) *Entry and exit rates*: Ψ_t^+ and Ψ_t^- are finite measures, $G_t^-(\mathbb{R}) \to 0$, and

$$\int \mathrm{IF}_\nu(x; G_t)\, \mathrm{d}G_t^-(x) = \Delta t \int \mathrm{IF}_\nu(x; G_t)\, \mathrm{d}\Psi_t^-(x) + o(\Delta t),$$
$$\int \mathrm{IF}_\nu(x; G_t)\, \mathrm{d}G_{t+\Delta t}^+(x) = \Delta t \int \mathrm{IF}_\nu(x; G_t)\, \mathrm{d}\Psi_t^+(x) + o(\Delta t).$$

(iii) *Exit and shocks*: exit in $(t, t+\Delta t]$ is independent of the shocks ε_{it} over that interval given positions at t, and, as usual with a continuum of units whose shocks are independent, cross-sectional averages equal expectations.

Condition (i) is the chain rule, which Definition 1 alone does not imply, applied to each step of the decomposition. Condition (ii) says that, over a short interval, exiters have measure $\Psi_t^- \Delta t$ and entrants $\Psi_t^+ \Delta t$. Condition (iii) says that who exits says nothing about the shocks that units would have received.

Assumption A2 (Smoothness)**.** One of the following two conditions holds; each result specifies which.

(a) *Smooth influence function.* $\partial_x \mathrm{IF}_\nu(x; G)$ and $\partial_{xx} \mathrm{IF}_\nu(x; G)$ exist and are jointly continuous in (x, G), with measures compared in total variation, on an open interval that contains positions at t and $t + \Delta t$. With $x' \equiv x + \mu_t(x)\Delta t + \sigma_t(x)\sqrt{\Delta t} \cdot \varepsilon$ the position at $t +$

Δt of a unit at position x at t and shock ε, the function $|\mu_t(x)\partial_x \mathrm{IF}_\nu(x;G_t)| + (\mu_t(x)^2 + \sigma_t(x)^2)|\partial_{xx}\mathrm{IF}_\nu(x;G_t)|$ is integrable against G_t, and the remainder of the second-order expansion of $\mathrm{IF}_\nu(\cdot;G_t)$ is dominated: there are $\delta > 0$ and a function $Y(x,\varepsilon) \geq 0$ with $\int \mathrm{E}[Y(x,\varepsilon_{it})]\,\mathrm{d}G_t(x) < \infty$ such that, for $\Delta t \leq \delta$,

$$\left|\mathrm{IF}_\nu(x';G_t) - \mathrm{IF}_\nu(x;G_t) - \partial_x \mathrm{IF}_\nu(x;G_t)(x'-x) - \frac{1}{2}\partial_{xx}\mathrm{IF}_\nu(x;G_t)(x'-x)^2\right| \leq \Delta t Y(x,\varepsilon).$$

(b) *Smooth densities.* For s near t, G_s has a density g_s, and Ψ_s^+ and Ψ_s^- have densities ψ_s^+ and ψ_s^-. The density solves the Kolmogorov forward equation of the diffusion of Assumption 1,

$$\partial_s g_s(x) = -\partial_x\left(\mu_s(x)g_s(x)\right) + \partial_{xx}\left(\frac{1}{2}\sigma_s(x)^2 g_s(x)\right) + \psi_s^+(x) - \psi_s^-(x),$$

with $\mu_s(x)g_s(x)$ continuously differentiable and $\sigma_s(x)^2 g_s(x)$ twice continuously differentiable in x. The influence function $\mathrm{IF}_\nu(x;G_t)$ is twice continuously differentiable in x except at finitely many points $q_1 < \cdots < q_K$, at which $\mathrm{IF}_\nu(x;G_t)$ and $\partial_x \mathrm{IF}_\nu(x;G_t)$ have one-sided limits. Finally, $|\mathrm{IF}_\nu(x;G_t)\partial_s g_s(x)| \leq \phi_0(x)$ for s near t, for some ϕ_0 with $\int \phi_0\,\mathrm{d}x < \infty$, and the boundary terms vanish: at the ends of the support of g_t,

$$\mathrm{IF}_\nu(x;G_t)\mu_t(x)g_t(x) \to 0, \qquad \mathrm{IF}_\nu(x;G_t)\partial_x\left(\sigma_t(x)^2 g_t(x)\right) \to 0,$$
$$\partial_x \mathrm{IF}_\nu(x;G_t)\sigma_t(x)^2 g_t(x) \to 0.$$

Assumption A2(a) covers discrete distributions, for statistics with smooth influence functions; its domination condition holds, for instance, when $\partial_{xx}\mathrm{IF}_\nu$ is bounded, as for the variance and the HHI. Assumption A2(b) covers all statistics in this paper, including those whose influence function has kinks (the top share) or jumps (quantiles). Under Assumption A2(b), $\partial_x \mathrm{IF}_\nu$ and $\partial_{xx}\mathrm{IF}_\nu$ in Proposition 2 are derivatives in the sense of distributions. Write $[h]_q \equiv h(q^+) - h(q^-)$ for the jump of a function h at q, and δ_q' for the derivative of the point mass at q, which acts on h as $\int h\,\mathrm{d}\delta_q' = -h'(q)$. The first derivative adds $\sum_k [\mathrm{IF}_\nu]_{q_k}\delta_{q_k}$ to the derivative away from the q_k; the second adds $\sum_k [\partial_x \mathrm{IF}_\nu]_{q_k}\delta_{q_k} + \sum_k [\mathrm{IF}_\nu]_{q_k}\delta_{q_k}'$. In the proofs, $\partial_x \mathrm{IF}_\nu$ and $\partial_{xx}\mathrm{IF}_\nu$ denote the derivatives away from the q_k; the jump terms are written out.

The second part of Proposition 2 also needs conditions on the weights of drift, which are evaluated at the distribution of continuing units, $\partial_x \mathrm{IF}_\nu(x_{it};G_t^C)$, rather than at G_t.

Assumption A3 (Weights)**.** The following two conditions hold.

(i) The weights at the distribution of continuing units are dominated: there are $\delta > 0$ and a function ϕ_1 integrable against G_t such that, for $\Delta t \leq \delta$,

$$|\partial_x \mathrm{IF}_\nu(x;G_t^C)|(|\mu_t(x)| + \sigma_t(x)) \leq \phi_1(x).$$

(ii) $\mathrm{IF}_\nu(x;G_t)$ is continuous in x, and $\partial_x \mathrm{IF}_\nu(x;G_t^C) \to \partial_x \mathrm{IF}_\nu(x;G_t)$ as $\Delta t \to 0$ for G_t-almost every x.

Condition (i) says that units' weights, times the size of their changes, are integrable uniformly over short intervals, so that drift per unit of time stays bounded. Condition (ii) says that removing the exiters from the distribution barely moves the weights of continuing units. It holds automatically under Assumption A2(a), and under Assumption A2(b) when the influence function has kinks but no jumps, as for top shares; it fails for quantiles, whose weight is a point mass, so that their drift is not defined in panel data.

Proof. First part: the law of motion. The proof first freezes the influence function at date t: by Assumption A1(i), the statistic changes, up to $o(\Delta t)$, by the change in the average of $\mathrm{IF}_\nu(x; G_t)$, so

$$\frac{\mathrm{d}\nu(G_t)}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int \mathrm{IF}_\nu(x; G_t)\, \mathrm{d}(G_{t+\Delta t} - G_t)(x), \tag{A1}$$

whenever the right side exists. The rest of the proof computes this change in the average of a fixed function, which comes from the movement of continuing units and from entry and exit. Let $\mathcal{L}_t \mathrm{IF}_\nu(x; G_t) \equiv \mu_t(x)\partial_x \mathrm{IF}_\nu(x; G_t) + \frac{1}{2}\sigma_t(x)^2 \partial_{xx} \mathrm{IF}_\nu(x; G_t)$ denote the generator of the diffusion of Assumption 1 applied to the influence function. It remains to show that

$$\begin{aligned} \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int \mathrm{IF}_\nu(x; G_t)\, \mathrm{d}(G_{t+\Delta t} - G_t)(x) = \int \mathcal{L}_t \mathrm{IF}_\nu(x; G_t)\, \mathrm{d}G_t(x) \\ + \int \mathrm{IF}_\nu(x; G_t)(\mathrm{d}\Psi_t^+(x) - \mathrm{d}\Psi_t^-(x)), \end{aligned} \tag{A2}$$

since the first term is the sum of the drift and displacement integrals of Proposition 2, and the second is the demography integral. I prove (A2) below under each version of Assumption A2.

Under Assumption A2(a). This route expands the influence function to second order in each continuing unit's change, as in Itô's lemma. It first separates entry and exit: by $G_t = G_t^C + G_t^-$, $G_{t+\Delta t} = G_{t+\Delta t}^C + G_{t+\Delta t}^+$, and Assumption A1(ii),

$$\begin{aligned} \int \mathrm{IF}_\nu(x; G_t)\, \mathrm{d}(G_{t+\Delta t} - G_t)(x) = \int \mathrm{IF}_\nu(x; G_t)\, \mathrm{d}(G_{t+\Delta t}^C - G_t^C)(x) \\ + \Delta t \int \mathrm{IF}_\nu(x; G_t)(\mathrm{d}\Psi_t^+(x) - \mathrm{d}\Psi_t^-(x)) + o(\Delta t). \end{aligned}$$

It remains to show that the term for continuing units divided by Δt converges to $\int \mathcal{L}_t \mathrm{IF}_\nu(x; G_t)\, \mathrm{d}G_t(x)$. By Assumptions 1 and A1(iii), a unit at position x at t is at x' at $t + \Delta t$, with ε distributed as ε_{it}, so

$$\frac{1}{\Delta t} \int \mathrm{IF}_\nu(x; G_t)\, \mathrm{d}(G_{t+\Delta t}^C - G_t^C)(x) = \int h_{\Delta t}\, \mathrm{d}G_t^C, \qquad h_{\Delta t}(x) \equiv \frac{\mathrm{E}[\mathrm{IF}_\nu(x'; G_t) - \mathrm{IF}_\nu(x; G_t)]}{\Delta t}.$$

Write $\mathrm{IF}_\nu(x'; G_t) - \mathrm{IF}_\nu(x; G_t) = \partial_x \mathrm{IF}_\nu(x; G_t)(x' - x) + \frac{1}{2}\partial_{xx} \mathrm{IF}_\nu(x; G_t)(x' - x)^2 + R_{\Delta t}(x, \varepsilon)$. Since $\mathrm{E}[\varepsilon_{it}] = 0$ and $\mathrm{E}[\varepsilon_{it}^2] = 1$, $\mathrm{E}[x' - x] = \mu_t(x)\Delta t$ and $\mathrm{E}[(x' - x)^2] = \sigma_t(x)^2 \Delta t + \mu_t(x)^2 \Delta t^2$, so

$$h_{\Delta t}(x) = \mathcal{L}_t \mathrm{IF}_\nu(x; G_t) + \frac{1}{2}\partial_{xx} \mathrm{IF}_\nu(x; G_t)\mu_t(x)^2 \Delta t + \frac{\mathrm{E}[R_{\Delta t}(x, \varepsilon)]}{\Delta t}.$$

The remainder vanishes faster than Δt. By Taylor's theorem, $R_{\Delta t}(x, \varepsilon) = \frac{1}{2}[\partial_{xx} \mathrm{IF}_\nu(y; G_t) - \partial_{xx} \mathrm{IF}_\nu(x; G_t)](x' - x)^2$ for some y between x and x'. For each ε, as $\Delta t \to 0$, $x' \to x$, so

$\partial_{xx}\mathrm{IF}_\nu(y;G_t) \to \partial_{xx}\mathrm{IF}_\nu(x;G_t)$ by continuity, while $(x'-x)^2/\Delta t = (\mu_t(x)\sqrt{\Delta t} + \sigma_t(x)\varepsilon)^2$ stays bounded; hence $R_{\Delta t}(x,\varepsilon)/\Delta t \to 0$. Since $|R_{\Delta t}|/\Delta t \le Y$ by Assumption A2(a), dominated convergence gives $\mathrm{E}[R_{\Delta t}(x,\varepsilon)]/\Delta t \to 0$, so $h_{\Delta t}(x) \to \mathcal{L}_t\mathrm{IF}_\nu(x;G_t)$ for G_t-almost every x.

It remains to integrate this pointwise limit against G_t^C, which differs from G_t by the exiters' mass. Write $\int h_{\Delta t}\,\mathrm{d}G_t^C = \int h_{\Delta t}\,\mathrm{d}G_t - \int h_{\Delta t}\,\mathrm{d}G_t^-$. For $\Delta t \le \min(\delta,1)$, $|h_{\Delta t}(x)| \le \phi_2(x) \equiv |\mu_t(x)\partial_x\mathrm{IF}_\nu(x;G_t)| + (\mu_t(x)^2+\sigma_t(x)^2)|\partial_{xx}\mathrm{IF}_\nu(x;G_t)| + \mathrm{E}[Y(x,\varepsilon)]$, which is integrable against G_t by Assumption A2(a). The integral against G_t therefore converges to $\int \mathcal{L}_t\mathrm{IF}_\nu(x;G_t)\,\mathrm{d}G_t(x)$ by dominated convergence. The integral against the exiters vanishes. Since $G_t^- \le G_t$, it is bounded, for any $K>0$, by $\int \phi_2\,\mathrm{d}G_t^- \le \int_{\{\phi_2>K\}} \phi_2\,\mathrm{d}G_t + KG_t^-(\mathbb{R})$. As $\Delta t \to 0$, the last term vanishes by Assumption A1(ii); the first is arbitrarily small for K large, since ϕ_2 is integrable. This proves (A2).

Under Assumption A2(b). This route integrates by parts against the forward equation, which moves the derivatives from the density onto the influence function. Since $|\mathrm{IF}_\nu(x;G_t)\partial_s g_s(x)| \le \phi_0(x)$ for s near t, the mean value theorem and dominated convergence give

$$\frac{1}{\Delta t}\int \mathrm{IF}_\nu(x;G_t)(g_{t+\Delta t}(x)-g_t(x))\,\mathrm{d}x \to \int \mathrm{IF}_\nu(x;G_t)\partial_t g_t(x)\,\mathrm{d}x.$$

Substitute the forward equation and integrate by parts between the kinks and the ends of the support of g_t. On each interval, omitting the arguments x and G_t, $-\int \mathrm{IF}_\nu\partial_x(\mu_t g_t) = -[\mathrm{IF}_\nu\mu_t g_t] + \int \partial_x\mathrm{IF}_\nu\mu_t g_t$ and $\int \mathrm{IF}_\nu\partial_{xx}(\frac{1}{2}\sigma_t^2 g_t) = [\mathrm{IF}_\nu\partial_x(\frac{1}{2}\sigma_t^2 g_t)] - [\partial_x\mathrm{IF}_\nu\frac{1}{2}\sigma_t^2 g_t] + \int \partial_{xx}\mathrm{IF}_\nu\frac{1}{2}\sigma_t^2 g_t$, where brackets denote the difference between the values at the two ends of the interval. The terms at the ends of the support vanish by Assumption A2(b). Since $\mu_t g_t$, $\sigma_t^2 g_t$, and $\partial_x(\sigma_t^2 g_t)$ are continuous, the terms at each q_k add up to the jump terms in the following display. Hence

$$\begin{aligned}\int \mathrm{IF}_\nu\partial_t g_t\,\mathrm{d}x = &\int \mathcal{L}_t\mathrm{IF}_\nu g_t\,\mathrm{d}x + \int \mathrm{IF}_\nu(\psi_t^+ - \psi_t^-)\,\mathrm{d}x \\ &+ \sum_k \left([\mathrm{IF}_\nu]_{q_k}\mu_t g_t + [\partial_x\mathrm{IF}_\nu]_{q_k}\frac{1}{2}\sigma_t^2 g_t - [\mathrm{IF}_\nu]_{q_k}\partial_x\left(\frac{1}{2}\sigma_t^2 g_t\right)\right)(q_k).\end{aligned}$$

With the distributional derivatives defined above, the first term and the sum are the drift and displacement integrals of Proposition 2, and the second term is the demography integral, which is (A2). For the top share, the influence function is continuous with a kink at the threshold, so only the term in $[\partial_x\mathrm{IF}_\nu]$ appears; for quantiles, it jumps at the quantile, and the terms in $[\mathrm{IF}_\nu]$ require the density to be continuously differentiable there.

Second part: the convergence of the accounting components. The proof shows that demography and drift converge to their integrals; displacement, the residual, then converges by the first part. Consider first demography. Exit removes the exiters' mass G_t^- from G_t, and entry adds the entrants' mass $G_{t+\Delta t}^+$ to $G_{t+\Delta t}^C$. By Assumption A1(i), each step changes the statistic, to first order, by the frozen influence function integrated against the mass removed or added: exit lowers the statistic by $\int \mathrm{IF}_\nu(x;G_t)\,\mathrm{d}G_t^-(x)$ and entry raises it by $\int \mathrm{IF}_\nu(x;G_t)\,\mathrm{d}G_{t+\Delta t}^+(x)$, up to $o(\Delta t)$. By Assumption A1(ii), these integrals are Δt times the same integrals against Ψ_t^- and Ψ_t^+, up to $o(\Delta t)$. Hence $\mathrm{Demography}_{t,t+\Delta t}/\Delta t$ converges to the demography integral.

Consider next drift. Since each unit's weight depends only on its initial position, drift

can replace each unit's change by its expected change given that position. By Assumptions 1 and A1(iii), a unit at position x at t has an expected change of $\mu_t(x)\Delta t$. For $\Delta t \leq 1$, $|\Delta x_{it}| \leq |\mu_t(x_{it})| + \sigma_t(x_{it})|\varepsilon_{it}|$, so the bound ϕ_1 of Assumption A3(i) makes $\partial_x \mathrm{IF}_\nu(x_{it}; G_t^C)\Delta x_{it}$ integrable. The law of iterated expectations then gives, exactly,

$$\begin{aligned}\frac{\mathrm{Drift}_{t,t+\Delta t}}{\Delta t} &= \int \partial_x \mathrm{IF}_\nu(x; G_t^C)\mu_t(x)\,\mathrm{d}G_t^C(x) \\ &= \int \partial_x \mathrm{IF}_\nu(x; G_t^C)\mu_t(x)\,\mathrm{d}G_t(x) - \int \partial_x \mathrm{IF}_\nu(x; G_t^C)\mu_t(x)\,\mathrm{d}G_t^-(x).\end{aligned}$$

The first integral converges to $\int \partial_x \mathrm{IF}_\nu(x; G_t)\mu_t(x)\,\mathrm{d}G_t(x)$, the drift integral, by the convergence of the weights, Assumption A3(ii), and dominated convergence with the bound ϕ_1. The second is bounded by $\int \phi_1\,\mathrm{d}G_t^-$, which vanishes as in the proof of (A2).

Finally, the three components sum to $\nu(G_{t+\Delta t}) - \nu(G_t)$, which divided by Δt converges to the sum of the three integrals by the first part. Since drift and demography converge to their integrals, $\mathrm{Displacement}_{t,t+\Delta t}/\Delta t$ converges to the displacement integral. □

What makes displacement first order in time is the quadratic variation of units' paths, not their differentiability: along a continuous path of finite variation, squared changes summed over ever shorter intervals vanish, and so does displacement. When neither version of Assumption A2 holds, as for the top p share among N discrete firms, the accounting of Proposition 1 still applies: it only requires the partial derivatives $\partial\nu(G_t^C)/\partial x_{it}$ at the observed positions.

Online Appendix

B Theory Appendix

This appendix contains the other theoretical results of Sections 1 and 2, which hold for any statistic. Appendix B.1 extends Proposition 2 to statistics of several variables, Appendix B.2 to processes with jumps, and Appendix B.3 to common and granular shocks. Appendix B.4 compares the accounting over an interval with the integral of the instantaneous rates: they differ by terms of second order in time, which vanish when the accounting is chained over actual dates. Appendix B.4 shows that chaining along a path interpolated between two dates, rather than over actual dates, removes all of displacement, with weights from economies that never existed. In each extension, drift keeps each unit's weight at its initial value, and displacement remains the nonlinear remainder of the statistic along realized changes.

B.1 Multivariate Statistics

The decomposition extends to statistics of the joint distribution of several variables $\boldsymbol{x} = [x_1, \ldots, x_n] \in \mathbb{R}^n$, such as average productivity weighted by size. The influence function is defined as with one variable,

$$\mathrm{IF}_\nu(\boldsymbol{x}; G) \equiv \lim_{h \downarrow 0} \frac{\nu(G + h\delta_{\boldsymbol{x}}) - \nu(G)}{h}.$$

Assume that the variables of continuing units follow $\mathrm{d}\boldsymbol{x}_{it} = \boldsymbol{\mu}_t(\boldsymbol{x}_{it})\,\mathrm{d}t + \boldsymbol{\Sigma}_t(\boldsymbol{x}_{it})\,\mathrm{d}\boldsymbol{W}_{it}$, where $\boldsymbol{\mu}_t$ is an n-dimensional vector, $\boldsymbol{\Sigma}_t$ an $n \times n$ matrix, and $\boldsymbol{W}_{it}$ an n-dimensional vector of standard Brownian motions, independent across units; the shocks to a unit's variables can be correlated. Units exit with measure $\Psi_t^- \Delta t$ and enter with measure $\Psi_t^+ \Delta t$ over a short interval, where $\Psi_t^\pm$ are now finite measures on $\mathbb{R}^n$. Then, under Assumptions A1 and A2(b) in their multivariate version,

$$\lim_{\Delta t \to 0} \frac{\Delta \nu}{\Delta t} = \underbrace{\int (\nabla \mathrm{IF}_\nu)' \boldsymbol{\mu}_t \,\mathrm{d}G_t}_{\text{Drift}} + \underbrace{\frac{1}{2} \int \mathrm{Tr}(\boldsymbol{\Sigma}_t' (\mathrm{Hess}\mathrm{IF}_\nu) \boldsymbol{\Sigma}_t)\,\mathrm{d}G_t}_{\text{Displacement}} + \underbrace{\int \mathrm{IF}_\nu (\mathrm{d}\Psi_t^+ - \mathrm{d}\Psi_t^-)}_{\text{Demography}}. \tag{B1}$$

The proof follows that of Proposition 2 under Assumption A2(b): substitute the multivariate Kolmogorov forward equation, $\partial_t g_t = -\sum_j \partial_{x_j}(\mu_{jt} g_t) + \frac{1}{2} \sum_{j,k} \partial_{x_j x_k}([\boldsymbol{\Sigma}_t \boldsymbol{\Sigma}_t']_{jk} g_t) + \psi_t^+ - \psi_t^-$, into (A1), and integrate by parts, once for the drift terms and twice for the diffusion terms. Kinks of IF_ν along finitely many smooth surfaces make the gradient and the Hessian derivatives in the sense of distributions.

B.2 Jumps

Proposition 2 assumes the diffusion of Assumption 1, under which only the first two local moments of changes in position matter at order Δt. I now allow for jumps, assuming that no unit enters or exits; with entry and exit, demography would be that of Proposition 2, unchanged.

Over a short interval Δt, a unit's position changes by

$$\Delta x_{it} = \mu_t(x_{it})\Delta t + \sigma_t(x_{it})\sqrt{\Delta t}\cdot\varepsilon_{it} + x_{it}J_{it}\Delta N_{it}, \tag{B2}$$

where N_{it} is a Poisson process with intensity $\lambda_t(x_{it})$ and J_{it} has distribution $F_J(\cdot; x_{it})$. For a unit at position x at t, with shock ε and one jump of size j, write $x'_j \equiv x + \mu_t(x)\Delta t + \sigma_t(x)\sqrt{\Delta t}\cdot\varepsilon + xj$ for its position at $t + \Delta t$.

Assumption B1 (Regularity of Jumps)**.** The following four conditions hold.

(i) *Bounded intensity*: λ_t is bounded.

(ii) *Finite jump terms*: $x\lambda_t(x)\mathrm{E}[|J_{it}| \mid x_{it} = x]|\partial_x \mathrm{IF}_\nu(x; G_t)|$ is integrable against G_t, and, for G_t-almost every x,

$$\int_{\mathbb{R}} |\mathrm{IF}_\nu(x(1+j); G_t) - \mathrm{IF}_\nu(x; G_t) - xj\partial_x \mathrm{IF}_\nu(x; G_t)|\,\mathrm{d}F_J(j; x) < \infty.$$

(iii) *Dominated jumps*: there are $\delta > 0$, a function $Y_J(x, \varepsilon, j) \geq 0$, and a function ϕ_J integrable against G_t such that, for $\Delta t \leq \delta$,

$$|\mathrm{IF}_\nu(x'_j; G_t) - \mathrm{IF}_\nu(x; G_t)| \leq Y_J(x, \varepsilon, j) \quad \text{and} \quad \lambda_t(x)\int_{\mathbb{R}} \mathrm{E}[Y_J(x, \varepsilon_{it}, j)]\,\mathrm{d}F_J(j; x) \leq \phi_J(x).$$

(iv) *Negligible multiple jumps*: the contribution of two or more jumps, $\mathrm{E}[(\mathrm{IF}_\nu(x_{i,t+\Delta t}; G_t) - \mathrm{IF}_\nu(x; G_t))\mathbf{1}_{\Delta N_{it} \geq 2} \mid x_{it} = x]$, is $o(\Delta t)$ for G_t-almost every x and bounded in absolute value by $\Delta t\phi_J(x)$.

Proposition B1. *Suppose that there is no entry or exit, that positions follow* (B2)*, and that Assumptions B1, A1, and A2(a) (smooth influence function) hold. Then*

$$\begin{aligned}\lim_{\Delta t \to 0} \frac{\Delta \nu}{\Delta t} = &\underbrace{\int \partial_x \mathrm{IF}_\nu(x)\left(\mu_t(x) + x\lambda_t(x)\mathrm{E}[J_{it} \mid x_{it} = x]\right)\mathrm{d}G_t(x)}_{\text{Drift}} \\ &+ \underbrace{\frac{1}{2}\int \partial_{xx}\mathrm{IF}_\nu(x)\sigma_t(x)^2\,\mathrm{d}G_t(x)}_{\text{Displacement (diffusive part)}} \\ &+ \underbrace{\int \lambda_t(x)\mathrm{E}\left[\mathrm{IF}_\nu(x(1+J_{it})) - \mathrm{IF}_\nu(x) - xJ_{it}\partial_x \mathrm{IF}_\nu(x) \;\middle|\; x_{it} = x\right]\mathrm{d}G_t(x)}_{\text{Displacement (jump part)}}.\end{aligned} \tag{B3}$$

Proof of Proposition B1. The chain rule (A1) does not depend on the process, so, under Assumption A2(a), it remains to prove (A2) with the generator

$$\begin{aligned}\mathcal{L}_t \mathrm{IF}_\nu(x; G_t) = &\mu_t(x)\partial_x \mathrm{IF}_\nu(x; G_t) + \frac{1}{2}\sigma_t(x)^2\partial_{xx}\mathrm{IF}_\nu(x; G_t) \\ &+ \lambda_t(x)\int_{\mathbb{R}} [\mathrm{IF}_\nu(x(1+j); G_t) - \mathrm{IF}_\nu(x; G_t)]\,\mathrm{d}F_J(j; x).\end{aligned}$$

Condition on the position x at t. If no jump occurs, which has probability $1 - \lambda_t(x)\Delta t + o(\Delta t)$, the expansion of the proof of Proposition 2 gives the first two terms of $\mathcal{L}_t \mathrm{IF}_\nu(x; G_t)$, times Δt. If one jump occurs, which has probability $\lambda_t(x)\Delta t + o(\Delta t)$, then $\mathrm{IF}_\nu(x'_j; G_t) \to \mathrm{IF}_\nu(x(1+j); G_t)$ for each ε and j, by continuity of the influence function, and dominated convergence, by Assumption B1(iii), gives the last term, times Δt. Two or more jumps contribute $o(\Delta t)$ by Assumption B1(iv). So $h_{\Delta t}(x) \to \mathcal{L}_t \mathrm{IF}_\nu(x; G_t)$ for G_t-almost every x, and $|h_{\Delta t}(x)|$ is bounded by the bound ϕ_2 of the proof of Proposition 2 plus $2\phi_J(x)$, which is integrable against G_t. Dominated convergence gives (A2) without the entry and exit terms. Finally, adding and subtracting $xj\partial_x \mathrm{IF}_\nu(x; G_t)$ inside the jump integral splits the jump term into the mean jump, $\lambda_t(x)\mathrm{E}[J_{it} \mid x_{it} = x] x \partial_x \mathrm{IF}_\nu(x; G_t)$, which belongs in drift, and the nonlinear remainder, the jump part of displacement. □

The decomposition of Section 1.1 measures these terms with jumps as well. The expected change of a unit at position x is $(\mu_t(x) + x\lambda_t(x)\mathrm{E}[J_{it} \mid x_{it} = x])\Delta t$, which includes the mean jump. So, when $\mathrm{E}[|J_{it}|]$ is finite, drift divided by Δt converges to the drift term of (B3), and displacement, the rest of the change, to the sum of its diffusive and jump parts.

When term-by-term integration is valid, a Taylor expansion of the influence function along each jump splits displacement by the cumulants of growth. Over a short interval, the n-th cumulant of the growth rate $\Delta x_{it}/x_{it}$ of a unit at position x is $\kappa_{nt}(x)\Delta t + o(\Delta t)$ for $n \geq 2$, with $\kappa_{2t}(x) \equiv \sigma_t(x)^2/x^2 + \lambda_t(x)\mathrm{E}[J_{it}^2 \mid x_{it} = x]$ and $\kappa_{nt}(x) \equiv \lambda_t(x)\mathrm{E}[J_{it}^n \mid x_{it} = x]$ for $n \geq 3$, since the diffusion contributes to the cumulants of order three and higher only at order $\Delta t^{3/2}$. So

$$\text{Displacement} = \sum_{n=2}^{\infty} \frac{1}{n!} \int \partial_x^n \mathrm{IF}_\nu(x; G_t) x^n \kappa_{nt}(x) \, \mathrm{d}G_t(x),$$

whose terms of order two, three, and four load on the variance, the skewness, and the kurtosis of growth. The quadratic statistics (the average, the variance, the HHI, and GE(2)) have only the variance term. For GE(α), $x^n \partial_x^n \mathrm{IF}_\nu(x; G_t) = (\alpha - 2) \cdots (\alpha - n + 1)\tilde{x}^\alpha$ for $n \geq 3$, so, for $\alpha < 2$, which includes the Theil index and GE(0), negatively skewed shocks, such as rare large losses, make the third-order term positive. For a statistic whose influence function has a kink, such as a top share, the series does not apply; displacement is the exact form above.

With jumps, the trapezoid approximation of Section 1.1 no longer converges to displacement: along a jump, the average of the weights at its two ends differs at order one from the weight averaged along it. In terms of the series above, the approximation multiplies the term of order n by $n/2$, so it is exact for the variance term and for the quadratic statistics but puts one and a half times its weight on the skewness term and twice its weight on the kurtosis term.

B.3 Common and Granular Shocks

Proposition 2 assumes a continuum of units whose shocks are independent, so that a unit's weight moves only because the unit moves. This fails in two ways. A common shock moves all units at once, to different degrees: stock returns move the wealth of households with different portfolios, and such a shock does not average out across units (Chamberlain and Rothschild,

1983). And when some units are large, each one's own shock moves the aggregates (Gabaix, 2011). In both cases, the distribution itself moves randomly, and so does each unit's weight. I treat common shocks first, then large units, without entry and exit; with entry and exit, demography is that of Proposition 2 when entrants and exiters are small, and a large entrant at x changes the statistic by $\nu(G_t + \delta_x) - \nu(G_t)$, the change that measured demography already uses.

Assumption B2. Units are of J types, and G_{jt} is the measure of the positions of the units of type j, so that $G_t = \sum_j G_{jt}$. Each unit has a negligible weight in the statistic, but a measure G_{jt} can put positive mass on a point. The position of a unit i of type j follows

$$\mathrm{d}x_{it} = \mu_{jt}(x_{it})\,\mathrm{d}t + \sigma_{jt}(x_{it})\,\mathrm{d}W_{it} + b_{jt}(x_{it}) \cdot \mathrm{d}Z_t,$$

where the W_{it} are standard Brownian motions, independent across units, $Z_t = (Z_t^1, \ldots, Z_t^K)$ is a vector of standard Brownian motions independent of the W_{it}, with correlation matrix ρ_t, $b_{jt}(x)$ is a vector of exposures, and μ_{jt}, σ_{jt}, b_{jt}, and ρ_t are continuous.

Each W_{it} is an idiosyncratic shock and each Z_t^k a common shock, to which $b_{jt}^k(x)$ is the exposure of the units of type j at x. Let $\pi_{jt}(x)$ be the share of type j among the units at x, so that $\mathrm{d}G_{jt} = \pi_{jt}\,\mathrm{d}G_t$. As in Section 2.1, $\mu_t(x) = \sum_j \pi_{jt}(x)\mu_{jt}(x)$ and $\sigma_t(x)^2 = \sum_j \pi_{jt}(x)\sigma_{jt}(x)^2$. The average exposure at x and the variance rate of the deviations of exposures from it are

$$\mathrm{E}_t[b \mid x] = \sum_j \pi_{jt}(x) b_{jt}(x), \quad \mathrm{Var}_t(b \mid x) = \sum_j \pi_{jt}(x) \left(b_{jt}(x) - \mathrm{E}_t[b \mid x]\right)' \rho_t \left(b_{jt}(x) - \mathrm{E}_t[b \mid x]\right).$$

The mass of units at each position either moves together or splits. Its shift, the change common to the units at the position, is $\mu_t(x)\,\mathrm{d}t + \mathrm{E}_t[b \mid x] \cdot \mathrm{d}Z_t$. Let $\Sigma_t(x, y) \equiv \mathrm{E}_t[b \mid x]' \rho_t \mathrm{E}_t[b \mid y]$ be the covariance rate of the shifts of positions x and y, and, for $z \in \mathbb{R}^K$, let G_t^z denote the distribution in which each unit at x moves to $x + z \cdot \mathrm{E}_t[b \mid x]$.

When the distribution moves, units' weights move with it, through the second-order influence function, the influence function at y of the functional $G \mapsto \mathrm{IF}_\nu(x; G)$:

$$\mathrm{IF}_\nu^{(2)}(x, y; G) \equiv \lim_{h \downarrow 0} \frac{\mathrm{IF}_\nu(x; G + h\delta_y) - \mathrm{IF}_\nu(x; G)}{h}.$$

Its cross derivative, $\partial_{xy}\mathrm{IF}_\nu^{(2)}(x, y; G)$, says how the weight of a unit at x responds to moving mass at y.

Proposition B2. *Suppose that there is no entry or exit, that Assumption B2 holds, and that, near the positions,* IF_ν *is twice continuously differentiable in* x *and* $\mathrm{IF}_\nu^{(2)}$ *continuously differentiable in* x *and* y.

Then

$$\begin{aligned}\mathrm{d}\nu(G_t) = &\underbrace{\int \partial_x \mathrm{IF}_\nu(x; G_t)\,(\mu_t(x)\,\mathrm{d}t + \mathrm{E}_t[b \mid x]\cdot \mathrm{d}Z_t)\,\mathrm{d}G_t}_{\text{Drift}}\\ &+\underbrace{\frac{1}{2}\int \partial_{xx}\mathrm{IF}_\nu(x; G_t)\,(\sigma_t(x)^2\,\mathrm{d}t + \mathrm{Var}_t(b \mid x)\,\mathrm{d}t)\,\mathrm{d}G_t}_{\text{Displacement (spreading)}}\\ &+\underbrace{\frac{1}{2}\Big(\int \partial_{xx}\mathrm{IF}_\nu(x; G_t)\Sigma_t(x,x)\,\mathrm{d}t\,\mathrm{d}G_t + \iint \partial_{xy}\mathrm{IF}^{(2)}_\nu(x,y; G_t)\Sigma_t(x,y)\,\mathrm{d}t\,\mathrm{d}G_t(x)\,\mathrm{d}G_t(y)\Big)}_{\text{Displacement (comovement)}}.\end{aligned}$$

Proof of Proposition B2. Consider first finitely many units with masses m_i, so that $G_t = \sum_i m_i \delta_{x_{it}}$, and let $F(x_1, \ldots, x_N) \equiv \nu(\sum_i m_i \delta_{x_i})$. Moving unit i by h perturbs G_t by $m_i(\delta_{x_{it}+h} - \delta_{x_{it}})$, so

$$\partial_i F = m_i \partial_x \mathrm{IF}_\nu(x_{it}; G_t), \qquad \partial_{ii'} F = \mathbf{1}_{i=i'} m_i \partial_{xx}\mathrm{IF}_\nu(x_{it}; G_t) + m_i m_{i'} \partial_{xy}\mathrm{IF}^{(2)}_\nu(x_{it}, x_{i't}; G_t).$$

With σ_i and b_i the volatility and the exposures of unit i's type at its position, Itô's lemma gives, since idiosyncratic shocks are independent across units,

$$\mathrm{d}\nu(G_t) = \sum_i \partial_i F\,\mathrm{d}x_{it} + \frac{1}{2}\sum_i \partial_{ii} F \sigma_i^2\,\mathrm{d}t + \frac{1}{2}\sum_{i,i'} \partial_{ii'} F b_i' \rho_t b_{i'}\,\mathrm{d}t.$$

Now let the largest mass go to zero, the units of each type at a point of positive mass keeping their total mass there. In the first term, drift, the idiosyncratic part has variance rate $\sum_i (\partial_i F)^2 \sigma_i^2$, which vanishes with the largest mass; the rest converges to the integral, since $\mathrm{d}G_{jt} = \pi_{jt}\,\mathrm{d}G_t$. In the second term, the part in $\partial_{xx}\mathrm{IF}_\nu$ converges to the part of the spreading term in σ_t^2; the part in $\partial_{xy}\mathrm{IF}^{(2)}_\nu$ is bounded by the largest mass times a convergent sum, so it vanishes. In the third term, write each exposure as the average exposure at its position plus its deviation from it. The terms in the average exposures are $\sum_{k,l} \rho_{kl,t} \partial_{z_k z_l} \nu(G_t^z)|_{z=0}$, which converges to twice the comovement term; pairs of units at the same point of positive mass give the diagonal of its double integral. The cross terms between average exposures and deviations vanish, since $\partial_{ii'} F$ depends on units only through their positions and masses, and the deviations average zero among the units at each position. Of the terms in the deviations, those in $\partial_{xy}\mathrm{IF}^{(2)}_\nu$ vanish for the same reason; those in $\partial_{xx}\mathrm{IF}_\nu$ converge to the part of the spreading term in $\mathrm{Var}_t(b \mid x)$. □

Drift weights each unit's realized change by its weight, so it contains the shocks; its expected rate is the first term without $\mathrm{d}Z_t$. Displacement, the sum of the last two terms, is of order $\mathrm{d}t$ and contains no shock. The mass of units at each position can either split or move together: splitting gives the spreading term, moving together the comovement term. What separates the two is whether a shock averages out among the units at a position, not whether it is common or idiosyncratic. The spreading term adds the differences in exposure among the units at a position to the variance of Proposition 2; it is non-negative for a statistic that satisfies the transfer principle.

The comovement term is half the curvature of the statistic along the shift, $\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\nu(G_t^z)|_{z=0}\,\mathrm{d}t$

with a single common shock: it combines the shift of each position acting on its own weight and the shift of the other positions, which moves the weight at x. It has no sign in general. It is zero when the shift moves the statistic linearly, in particular without common shocks, in which case the proposition reduces to Proposition 2 with several types at each position. When the average exposure in growth rates, $\mathrm{E}_t[b \mid x]/x$, is the same at every position, the common shocks rescale the distribution, so they add no comovement term for a scale-invariant statistic.

Granularity. A large unit is one whose position is large relative to the aggregates the statistic depends on, so that its own shock moves them. With finitely many units, each of mass one, the Itô formula in the proof above holds exactly, with $m_i = 1$. Let unit i load, with exposure σ_{it}, on a shock Z_t^i of its own, the shocks of units i and k being correlated at the rate $\rho_{ik,t}$. A unit alone at its position cannot split, so its shock adds nothing to the spreading term and enters the comovement term through both parts: $\frac{1}{2}\partial_{xx}\mathrm{IF}_\nu(x_{it}; G_t)\sigma_{it}^2$, the unit's shift acting on its own weight, and $\frac{1}{2}\partial_{xy}\mathrm{IF}_\nu^{(2)}(x_{it}, x_{it}; G_t)\sigma_{it}^2$, the granular term: the unit's shock moves the distribution, which moves its own weight. Relative to the first part, the granular term is of the order of the unit's share of the aggregate: for the HHI of sales, the ratio is $3\mathrm{HHI}_t - 4s_{it}$. It vanishes as the largest share goes to zero, but not as the number of units grows when the size distribution is fat-tailed enough that the sum of squared shares does not vanish. The shocks of large units also add to the random part of drift a variance rate

$$\sum_{i,k} \partial_x \mathrm{IF}_\nu(x_{it}; G_t)\partial_x \mathrm{IF}_\nu(x_{kt}; G_t)\rho_{ik,t}\sigma_{it}\sigma_{kt},$$

the squared granular volatility of the statistic (Gabaix, 2011). With large units, the proposition rules out rank-based statistics such as top shares and the Gini, whose weights jump when two units swap ranks; the estimator of Proposition 1 and the split below still apply to them.

Measuring the terms. Drift over an interval is the Itô sum over that interval, so it contains the shocks. Chained over ever finer partitions, drift converges to the Itô integral and displacement to the integral of the displacement terms. Over a single short interval, the random part of drift is of order $\sqrt{\Delta t}$, so it dominates the variance of the change in the statistic, and Proposition 2 holds for expectations. At any horizon, the split groups units at the same initial position. Let $M(x)$ be the mean change of units at initial position x, $x_{it}^m \equiv x_{it} + M(x_{it})$ their positions after this mean change, and G^m their distribution. Then

$$\text{Displacement} = \underbrace{\nu(G^C_{t+\Delta t}) - \nu(G^m)}_{\text{Spreading}} + \underbrace{\nu(G^m) - \nu(G^C_t) - \sum_{i \in C} \frac{\partial \nu(G_t^C)}{\partial x_{it}} \Delta x_{it}}_{\text{Comovement}}.$$

Units at the same initial position have the same weight, and their deviations from $M(x)$ average zero, so drift is the drift of the mean changes, and comovement is their nonlinear effect. Since $G^C_{t+\Delta t}$ is a mean-preserving spread of G^m, spreading is non-negative at any horizon under the transfer principle. A large unit alone at its position has no spreading. Over short intervals,

the terms converge to those of the proposition. In a sample with distinct positions, each unit is alone and all displacement is comovement; estimating mean changes over bins instead makes the split depend on the bins. Narrow bins leave some idiosyncratic changes in the estimated mean, while wide bins count differences in mean changes as spreading.

For an ordered weighted average, such as the Gini or a top share, a unit's weight depends on its position only through its rank, and drift holds ranks fixed. When the shift preserves the order of positions, it changes no weight, so the comovement term is zero and all of displacement is spreading: the re-ranking that comes from units' deviations from the shift of their position. A smooth shift preserves the order of positions over short intervals, so the common shocks of Proposition B2 move ordered weighted averages only through drift and spreading.

B.4 Second-Order Terms

This subsection derives the terms of second order in time by which the accounting of Proposition 1 over an interval differs from the integrals of the instantaneous rates of Proposition 2 (Section 2.2). These terms vanish when the accounting is chained over ever shorter intervals between actual dates.

Accounting at finer intervals. Without entry and exit, the accounting over $[t, t+\Delta t]$ gives $\Delta\nu(G_t) = \text{Drift}_{t,t+\Delta t} + \text{Displacement}_{t,t+\Delta t}$, and chaining it over the two halves splits the same change into the sums of the components over the halves. Whatever the process behind units' changes, the difference between the two splits has an exact form. Write $\omega_{is} \equiv \partial_x \text{IF}_\nu(x_{is}; G_s)$ for unit i's weight at s. Exactly,

$$\text{Drift}_{t,t+\Delta t} - \sum_{s\in\{t,t+\Delta t/2\}} \text{Drift}_{s,s+\Delta t/2} = -\sum_i (\omega_{i,t+\Delta t/2} - \omega_{it})(x_{i,t+\Delta t} - x_{i,t+\Delta t/2}),$$

and the same difference for displacement is its opposite. Drift over the whole interval weights units' changes over the second half by ω_{it}, whereas drift over the second half weights them by $\omega_{i,t+\Delta t/2}$. Under the setup below, this difference is $-\frac{1}{4}\Omega_t \Delta t^2 + o(\Delta t^2)$ (Proposition B3).

Continuous-time limit. When units follow diffusions at every date, integrating the law of motion of Section 2.1 decomposes the change over $[t, t+\Delta t]$ into the integrals of the instantaneous rates, $\int_t^{t+\Delta t} \text{Drift}_s \, ds + \int_t^{t+\Delta t} \text{Displacement}_s \, ds$. By Proposition 2, the gap between each component over the interval and the integral of its rate is $o(\Delta t)$; the proposition below gives it to second order, in Δt^2. These terms say how the split depends on the length of the interval, even when positions are measured without error.

Comparing intervals requires a single process observed at every length, in which the coefficients move with units within the interval. Consider a continuum of units of mass one, without entry and exit. Each unit has a type j, fixed over time, and follows $dx_{is} = \mu_{js}(x_{is})\, ds + \sigma_{js}(x_{is})\, dW_{is}$, with independent Brownian motions W_{is}. Write $\mu_{is} \equiv \mu_{js}(x_{is})$ and $\sigma_{is} \equiv \sigma_{js}(x_{is})$, with $\partial_x \mu_{is} \equiv \mu'_{js}(x_{is})$ and $\partial_{xx}\mu_{is} \equiv \mu''_{js}(x_{is})$. Inside $\text{E}[\cdot]$, the influence function and its derivatives are evaluated at $(x_{it}; G_t)$ and the coefficients at x_{it}. Drift over the interval is $\text{E}[\omega_{it}(x_{i,t+\Delta t} - x_{it})]$,

and the instantaneous rates are $\text{Drift}_s = \mathrm{E}[\omega_{is}\mu_{is}]$ and $\text{Displacement}_s = \frac{1}{2}\mathrm{E}[\sigma_{is}^2\partial_{xx}\mathrm{IF}_\nu(x_{is};G_s)]$. The influence function at a fixed position y changes at the rate $\partial_t\mathrm{IF}_\nu(y;G_t) = \mathrm{E}[\mu_{it}\partial_x\mathrm{IF}_\nu^{(2)}(x_{it},y;G_t) + \frac{1}{2}\sigma_{it}^2\partial_{xx}\mathrm{IF}_\nu^{(2)}(x_{it},y;G_t)]$, with $\mathrm{IF}_\nu^{(2)}$ the second-order influence function of Appendix B.3, its derivatives taken in its first argument; the weight at y changes at $\partial_t\partial_x\mathrm{IF}_\nu$.

Proposition B3 (Components over an Interval)**.** *Under the setup above, if* $\mathrm{IF}_\nu(x;G_s)$ *is three times continuously differentiable in* x,[24] *with* $\partial_x\mathrm{IF}_\nu(x;G_s)$ *continuously differentiable in* s*, if each* $\mu_{js}(x)$ *is twice continuously differentiable in* x*, and if expectations and integrals can be exchanged, then*

$$\text{Drift}_{t,t+\Delta t} = \int_t^{t+\Delta t}\text{Drift}_s\,\mathrm{d}s - \frac{1}{2}\Omega_t\Delta t^2 + o(\Delta t^2),$$
$$\text{Displacement}_{t,t+\Delta t} = \int_t^{t+\Delta t}\text{Displacement}_s\,\mathrm{d}s + \frac{1}{2}\Omega_t\Delta t^2 + o(\Delta t^2),$$

with

$$\Omega_t \equiv \mathrm{E}\left[\left(\mu_{it}^2 + \sigma_{it}^2\partial_x\mu_{it}\right)\partial_{xx}\mathrm{IF}_\nu + \frac{1}{2}\mu_{it}\sigma_{it}^2\partial_{xxx}\mathrm{IF}_\nu + \mu_{it}\partial_t\partial_x\mathrm{IF}_\nu\right].$$

Proof of Proposition B3. The stochastic integral in $x_{i,t+\Delta t} - x_{it}$ has mean zero given the unit's position and type at t, so drift over the interval is the integral of $\mathrm{E}[\omega_{it}\mu_{is}]$ over $s \in [t, t+\Delta t]$, and the integral of instantaneous drift minus drift over the interval is the integral of $\mathrm{E}[(\omega_{is} - \omega_{it})\mu_{is}]$. By Itô's formula,

$$\mathrm{d}\omega_{is} = \left(\mu_{is}\partial_{xx}\mathrm{IF}_\nu + \frac{1}{2}\sigma_{is}^2\partial_{xxx}\mathrm{IF}_\nu + \partial_s\partial_x\mathrm{IF}_\nu\right)\mathrm{d}s + \partial_{xx}\mathrm{IF}_\nu\sigma_{is}\,\mathrm{d}W_{is},$$

with the influence function evaluated at $(x_{is};G_s)$, and the martingale part of μ_{is} is $\partial_x\mu_{is}\sigma_{is}\,\mathrm{d}W_{is}$. Split the integrand into $\mathrm{E}[(\omega_{is} - \omega_{it})\mu_{it}]$ and $\mathrm{E}[(\omega_{is} - \omega_{it})(\mu_{is} - \mu_{it})]$. The first is $s - t$ times $\mathrm{E}[\mu_{it}(\mu_{it}\partial_{xx}\mathrm{IF}_\nu + \frac{1}{2}\sigma_{it}^2\partial_{xxx}\mathrm{IF}_\nu + \partial_t\partial_x\mathrm{IF}_\nu)]$, plus $o(s-t)$. The second is $s - t$ times the rate of the quadratic covariation of ω_{is} and μ_{is}, $\mathrm{E}[\sigma_{it}^2\partial_x\mu_{it}\partial_{xx}\mathrm{IF}_\nu]$, plus $o(s-t)$. The integrand is therefore $\Omega_t(s-t) + o(s-t)$, whose integral over the interval is $\frac{1}{2}\Omega_t\Delta t^2 + o(\Delta t^2)$. The gap for displacement is the opposite, since the two decompositions add up to the same change. □

When Ω_t is continuous in t and the remainders are uniform in t, applying the proposition to the interval and to each of K equal intervals within it gives

$$\text{Drift}_{t,t+\Delta t} = \sum_s \text{Drift}_{s,s+\Delta t/K} - \left(1 - \frac{1}{K}\right)\frac{1}{2}\Omega_t\Delta t^2 + o(\Delta t^2),$$

and the opposite for displacement: chaining removes a fraction $1 - 1/K$ of the second-order terms. As K grows, drift chained over finer intervals converges to the integral of its rate, not to the whole change: displacement remains, since its rate is first order in time. Displacement over an interval exceeds the integral of its rate by a fraction $b \equiv \Omega_t\Delta t/(2\cdot\text{Displacement}_t)$, so displacement per unit of time over H consecutive intervals of length Δt is about $(1 + Hb)\cdot$

[24] The proposition can be adapted to influence functions with kinks or jumps, such as those of top shares, when positions have a smooth density, as in Assumption A2(b): the derivatives of the influence function are then read as distributions, and integrating by parts moves them onto the density. For a top share, Ω_t then depends on the density and its slope at the threshold.

Displacement$_t$. Comparing intervals on the same units therefore measures these terms: they are small when the components per unit of time barely move with the interval, as in firms' sales (Figure 2).

Drift over an interval applies the initial weights to units' changes, whereas the integral of its rate updates the weights as units move. Ω_t is the rate at which the covariance between the change in a unit's weight and its subsequent expected change builds up. It has four sources: a unit's expected change moves its weight in the same direction ($\mu^2\partial_{xx}\mathrm{IF}_v$); a shock moves both a unit's weight and its expected change ($\sigma^2\partial_x\mu\partial_{xx}\mathrm{IF}_v$), which, with mean reversion and a convex influence function, makes drift over the interval exceed the integral of instantaneous drift; shocks move units toward positions with higher or lower weights when the curvature varies with position ($\frac{1}{2}\mu\sigma^2\partial_{xxx}\mathrm{IF}_v$); and the weight at each position changes as the distribution moves ($\mu\partial_t\partial_x\mathrm{IF}_v$). Unlike Proposition 2, Ω_t involves the dispersion of expected changes among the units at each position, through μ^2. For the exponential mean with $\mathrm{d}x_{is} = (\mu_i - \kappa x_{is})\,\mathrm{d}s + \sigma\,\mathrm{d}W_{is}$, $b = \Delta t(\mathrm{Var}^s(\mu_i - \kappa x_{it}) - \kappa\sigma^2)/\sigma^2$, where Var^s is the variance across units weighted by $e^{\gamma x_{it}}$: persistent differences in growth raise displacement over an interval; mean reversion lowers it.

Entry and exit. With entry and exit, all three components have further terms of second order in time. The accounting counts an entrant in entry at its final position and an exiter in exit at its initial position, whereas the rates count their changes in drift and displacement while they are present. For a population-invariant statistic, with entrants and exiters distributed like the units present, this moves nothing on average, and only the terms in Ω_t remain. Exiters, however, are selected on their changes: where exit is less likely at higher positions, units hit by bad shocks are more likely to exit next. The rates count this decline in drift until the unit exits, the shadow of death of Griliches and Regev (1995), whereas the accounting counts it in exit. With positive weights, drift over an interval therefore exceeds the integral of its rate, and demography falls short of its integral by the same amount; displacement is unaffected. Chaining over K equal intervals still removes a fraction $1 - 1/K$ of each of these terms.

Chaining along fictitious interpolated paths. With two dates only, the accounting can be chained along a path interpolated between them rather than over actual dates. Chaining over actual dates removes only the second-order terms of Proposition B3: as the intervals shrink, displacement converges to the integral of its rate, not to zero. Chaining along an interpolation removes all of displacement, with weights from economies that never existed.

Along the straight path from initial to final positions, unit i is at $x_i(u) = x_{it} + u\Delta x_{it}$ for $u \in [0,1]$; let $G^C(u)$ denote the distribution of continuing units along it.[25] The change among continuing units weights each unit's change by its average weight along the path, whereas

[25]This requires the statistic to be continuous along the path and differentiable except at finitely many points, as for top shares and the Gini.

drift weights it by its initial weight, so

$$\text{Displacement} = \sum_{i \in C} \left(\int_0^1 \frac{\partial \nu(G^C(u))}{\partial x_i(u)} \, \mathrm{d}u - \frac{\partial \nu(G_t^C)}{\partial x_{it}} \right) \Delta x_{it}. \tag{B4}$$

Dividing the path into K equal steps and evaluating each unit's weight at the start of each step gives the chained calculation

$$\text{Chain}_K \equiv \frac{1}{K} \sum_{k=0}^{K-1} \sum_{i \in C} \frac{\partial \nu(G^C(k/K))}{\partial x_i(k/K)} \Delta x_{it}.$$

For $K = 1$, it is drift. When $\nu(G^C(u))$ has a continuous derivative in u, it converges as $K \to \infty$ to the whole change among continuing units, $\nu(G^C_{t+\Delta t}) - \nu(G^C_t)$: an accounting identity along the interpolation, whose summands beyond the first evaluate weights in intermediate economies that never existed.

The interpolation absorbs displacement because it turns shocks into persistent growth. Along the straight path, each unit grows at the constant rate $\Delta x_{it}/\Delta t$ and faces no shocks, so its realized shock becomes a persistent difference in growth among units at the same position, the first source of Ω_t, which chaining removes. In the actual economy, the same dispersion comes from shocks that do not persist, so chaining over actual dates leaves it in displacement. When $\nu(G^C(u))$ is quadratic in u, as for the variance and the HHI, chaining along the interpolation removes the same fraction $1 - 1/K$ as chaining over actual dates, but of all of displacement: $\text{Chain}_K = \text{Drift} + (1 - 1/K) \cdot \text{Displacement}$. For a top share, a unit that enters the top group through a gain keeps gaining along the path after it crosses, so the rest of its gain counts as growth within the top, while a unit that leaves through a loss keeps losing outside it. Chaining along the interpolation thus treats shocks as success that predicts further success, even when every unit has the same expected change.

C Implementation Appendix

This appendix contains the results behind Section 3. Appendix C.1 treats measurement error and flows. Appendix C.2 treats the choice of the variable for positions, normalized or not and in levels or logs, and the choice between decomposing the change in a statistic and its growth (Appendix C.2).

C.1 Measurement Error and Flows

Measured positions can differ from the positions of Assumption 1 in two common ways: they can contain transitory errors, or they can be averages over a period rather than values at a date. In both cases, the measured position is the true one plus a gap, $x_{it}^{\text{obs}} = x_{it} + e_{it}$. This subsection derives the bias that a gap creates in drift and displacement, applies it to noise and to flows, and gives the correction with lagged weights and its limits.

Proposition C1 (Bias from a Gap)**.** *Suppose that the measured position is* $x_{it}^{\text{obs}} = x_{it} + e_{it}$*, and that (i) a unit's gap has the same mean at both dates; (ii) net of this mean, the gap at each date is independent of the unit's true position one interval earlier, of its earlier gaps, and of its true change over the following interval; (iii) gaps are independent across units; and (iv) the gap moves the statistic of continuing units, and that of all units, by the same amount at both dates. Then, in expectation, up to terms of third order in the gaps and the true changes, and up to terms in the mean of the gap times expected changes, the change among continuing units and demography are unaffected, and*

$$\begin{aligned}\text{Drift}_{t,t+\Delta t} - \text{Drift}^{\text{obs}}_{t,t+\Delta t} &= \text{Displacement}^{\text{obs}}_{t,t+\Delta t} - \text{Displacement}_{t,t+\Delta t} \\ &= \sum_{i \in C} \partial_{xx}\text{IF}_\nu(x_{it}; G_t^C)\left(\text{Var}(e_{it}) + \text{Cov}(e_{it}, x_{it} - x_{i,t-\Delta t})\right).\end{aligned}$$

The variance of the gap pushes drift down: a unit whose position is overstated looks large and falls back. A gap that falls short of a recent rise pushes drift up: the unit looks small and keeps rising. Neither term depends on the length of the interval.

Proof of Proposition C1. Write m_i for the mean of unit i's gap and $\tilde{e}_{it} \equiv e_{it} - m_i$. Since the weight is the slope of the influence function, $\partial\nu(G_t^C)/\partial x_{it} = \partial_x \text{IF}_\nu(x_{it}; G_t^C)$, the measured weight and the measured change are, to first order in the gap,

$$\frac{\partial \nu(G_t^C)}{\partial x_{it}} + \partial_{xx}\text{IF}_\nu(x_{it}; G_t^C)e_{it} \quad \text{and} \quad \Delta x_{it} + \tilde{e}_{i,t+\Delta t} - \tilde{e}_{it},$$

where the mean of the gap drops out of the change because it is the same at both dates. The gaps of other units also move the weight, through the distribution, but they are independent of the unit's own gaps and change, so their products with them average zero.

Besides the true weight times the true change, which is true drift, four products remain. The true weight times $\tilde{e}_{i,t+\Delta t}$ averages zero, since this gap is independent of the position at t. For the true weight times $-\tilde{e}_{it}$, write the weight as its value at the unit's position one interval earlier plus the change since, $\partial\nu(G_t^C)/\partial x_{it} \approx \partial_x \text{IF}_\nu(x_{i,t-\Delta t}; G_t^C) + \partial_{xx}\text{IF}_\nu(x_{it}; G_t^C)(x_{it} - x_{i,t-\Delta t})$. The first term is independent of $\tilde{e}_{it}$, so the product averages $-\partial_{xx}\text{IF}_\nu(x_{it}; G_t^C)\text{Cov}(e_{it}, x_{it} - x_{i,t-\Delta t})$. The gap in the weight times the true change averages $\partial_{xx}\text{IF}_\nu(x_{it}; G_t^C)m_i\text{E}[\Delta x_{it}]$, since $\tilde{e}_{it}$ is independent of the change that follows it: this is the mean of the gap times an expected change. The gap in the weight times $\tilde{e}_{i,t+\Delta t} - \tilde{e}_{it}$ averages $-\partial_{xx}\text{IF}_\nu(x_{it}; G_t^C)\text{Var}(e_{it})$, since $\tilde{e}_{i,t+\Delta t}$ is independent of e_{it}. The terms left out of the expansions are of third order in the gaps and the changes. Summing over units gives the bias in drift.

The change among continuing units is the change in the statistic of their measured positions, which the gap moves by the same amount at both dates, so it is that of true positions. Displacement, the rest of this change, therefore moves by minus the bias in drift. The gap also moves the statistic of all units by the same amount at both dates, so the total change and demography are unaffected. □

The proof requires a smooth influence function. For ordered weighted averages, the expected weight of a unit measured with error is smooth when positions have a smooth density

(Assumption A2(b)), and the proposition holds with the sum read as the integral of $\partial_{xx}\mathrm{IF}_\nu$ against the distribution of continuing units. For top p wealth, noise of variance σ_e^2 then lowers drift by $\sigma_e^2 g(q_{1-p,t})$, where g is the density of positions. Demography is unaffected, but entry and exit can still be biased one by one when the effect of the gap depends on the number of units, as for the HHI. With flows, a unit's first and last flows cover only part of a period, so I count these periods as entry and exit rather than as continuing.

Measurement error. Suppose that true positions follow the diffusion of Assumption 1, but the data record $x_{it} + e_{it}$, where the error e_{it} has mean zero and variance σ_e^2 and is independent of true positions, across units, and across dates. Then $\mathrm{Cov}(e_{it}, x_{it} - x_{i,t-\Delta t}) = 0$, so, by Proposition C1, noise moves $\sigma_e^2 \sum_{i \in C} \partial_{xx}\mathrm{IF}_\nu(x_{it}; G_t^C)$ from drift to displacement. Under the transfer principle, $\partial_{xx}\mathrm{IF}_\nu \geq 0$ (Section 2.1), so measured drift is too low and measured displacement too high. Displacement is proportional to the length of the interval while the bias is not, so the bias matters more over shorter intervals. For the variance, measured displacement is $\mathrm{Var}(\Delta x_{it}) + 2\sigma_e^2$ and measured drift $2\mathrm{Cov}(x_{it}, \Delta x_{it}) - 2\sigma_e^2$.

Since the bias does not grow with the interval, comparing intervals of different lengths shows whether it matters. Take the units observed at three dates, $t - \Delta t$, t, and $t + \Delta t$. When true changes over consecutive intervals are uncorrelated, displacement adds up across the two intervals, while the noise term enters once in each interval and once in the combined one. The sum of displacement over the two intervals minus displacement over the combined interval is therefore $\sigma_e^2 \sum_{i \in C} \partial_{xx}\mathrm{IF}_\nu(x_{it}; G_t^C)$: positive when noise matters, zero when it does not. Panel regressions use the same comparison of differences of different lengths (Griliches and Hausman, 1986).

Lagged weights. The correction replaces each unit's weight at t, which contains the error at t, by a prediction from the unit's earlier positions, which does not. The simplest prediction is the lagged weight of (3), the unit's weight at its position one interval earlier, with C the units present at the three dates. Lagged displacement is the rest of the change among continuing units. In the variance, lagged drift is $2\mathrm{Cov}(x^{\mathrm{obs}}_{i,t-\Delta t}, \Delta x^{\mathrm{obs}}_{it}) = 2\mathrm{Cov}(x_{it}, \Delta x_{it}) - 2\mathrm{Cov}(\Delta x_{i,t-\Delta t}, \Delta x_{it})$, true drift when true changes over consecutive intervals are uncorrelated.

Equivalently, the correction is the comparison of intervals above, for any statistic. Drift over the combined interval with the weight at $t - \Delta t$ uses the same weight over both intervals, so

$$\mathrm{Drift}_{t-\Delta t,t+\Delta t} = \mathrm{Drift}_{t-\Delta t,t} + \mathrm{Drift}^{\mathrm{lag}}_{t,t+\Delta t}.$$

The change among a fixed set of units adds up across intervals, so the correction, lagged drift minus drift over the second interval, is exactly the sum of displacement over the two intervals minus displacement over the combined interval. For the variance, lagged displacement is $\mathrm{Var}(\Delta x^{\mathrm{obs}}_{it}) + 2\mathrm{Cov}(\Delta x^{\mathrm{obs}}_{i,t-\Delta t}, \Delta x^{\mathrm{obs}}_{it})$, the correction that Zhou (1996) proposes for the realized variance of noisy high-frequency returns, with the autocovariance taken across units rather than over time. With serially correlated noise, this correction is biased and weights two intervals earlier are needed (Hansen and Lunde, 2006); the two-scale estimator of Zhang et al. (2005)

similarly compares realized variance at two sampling frequencies. Lagged weights remove the error term with no expansion in the error, for any statistic, ranks included, when errors have mean zero given the past and are independent across dates; interpreting the correction as the size of this bias uses the approximation of Proposition C1.

When only some continuing units are observed at three dates, the correction is estimated on them as a share of their displacement, $\chi \equiv (\text{Drift}^{\text{lag}} - \text{Drift})/\text{Displacement}$. The drift and displacement of all continuing units then become $\text{Drift} + \chi \cdot \text{Displacement}$ and $(1-\chi) \cdot \text{Displacement}$. Noise makes χ positive. Flows alone make it $-1/2$ over an interval of one flow period (below).

Flows. Suppose instead that positions are measured without error, but the data record flows, such as a firm's annual sales or a household's annual income: the average of the position over a period rather than its value at the end of the period. Time is measured in periods. Each unit's expected change and volatility, μ_i and σ_i, are constant from period k to period $k+H$, so that $\mathrm{d}x_i(s) = \mu_i \,\mathrm{d}s + \sigma_i \,\mathrm{d}W_i(s)$. The flow of period k is $X_{ik} = \int_{k-1}^{k} x_i(s)\,\mathrm{d}s$. Its gap is

$$e_{ik} = X_{ik} - x_i(k) = -\int_{k-1}^{k} (u - k + 1)\,\mathrm{d}x_i(u),$$

a weighted sum of the changes of period k, with weights rising from zero at its start to one at its end. Its mean, $-\mu_i/2$, is the same at all dates. Net of this mean, the gap at $k+H$ depends only on the shocks after $k+H-1 \geq k$. Its variance is $\sigma_i^2/3$ and its covariance with the unit's change over period k is $-\sigma_i^2/2$: the average lags behind a unit that rose during the period. By Proposition C1, flows move

$$\left(\frac{1}{2} - \frac{1}{3}\right) \sum_i \partial_{xx}\mathrm{IF}_\nu(x_i(k); G_k^C)\sigma_i^2 = \frac{1}{3} \cdot \text{Displacement}$$

from displacement to drift, whatever the length H of the interval, where Displacement is the rate of Proposition 2, per period. The lag creates momentum, which raises drift by Displacement; the variance of the gap, like noise, lowers it by two thirds of Displacement. For the variance and a random walk, this is the result of Working (1960) that changes in period averages of a random walk have two thirds of the variance of changes in its values.

The result survives mean reversion: for the variance and a stationary Ornstein–Uhlenbeck process that reverts at a rate $\kappa \leq 1$ per period, the bias lies between $\sigma^2/3 - \sigma^2\kappa/12$ and $\sigma^2/3$ at every horizon. It also survives taking the log of a flow, as for aggregate productivity (Section 4.1), since the difference from the average of the log position depends only on the unit's volatility, which drops out of its changes.

Since the shift does not depend on the length of the interval, displacement of flows over H periods is $(H - 1/3) \cdot \text{Displacement}$. In the units of measured displacement, flows alone thus produce a correction of $-1/2$ over one period and of $-1/(3H-1)$ over H periods, whereas the share of noise falls with the length of the interval. The comparison of intervals has the opposite sign from noise: displacement over H periods exceeds the sum of the H one-period

displacements by a fraction $(H-1)/(2H)$ of the sum, which is bounded by one half, whereas the excess from persistent differences in growth keeps growing with H (Appendix B.4). Lagged weights remove the bias from flows as well, provided the lag covers at least one period: for the variance and a random walk, lagged displacement of flows over one period is $2\sigma^2/3 + \sigma^2/3 = \sigma^2$.

Persistence and extrapolated weights. When true changes over consecutive intervals are correlated, lagged weights also change drift in error-free data, by minus the sum over units of the change in the unit's weight times its next change. For the process of Appendix B.4, the argument of Proposition B3 shows that weights τ intervals earlier change drift by $-\tau\Omega_t\Delta t^2$, a term of second order in time. Persistent differences in growth among units at the same position, as among young firms, make the correction negative: it moves part of drift into displacement. Mean reversion makes it positive. Both are true dynamics, which drift and displacement with current weights already split as Section 1.1 defines them, and a model measured with the same weights contains the same term (Appendix D.2.4). A longer lag moves more drift into displacement under persistence and more displacement into drift under mean reversion, so a correction that changes with the lag signals them.

Extrapolated weights remove this term. They predict each unit's weight at t by extrapolating it linearly from its two past values,

$$2\partial\nu(G^C_{t-\Delta t})/\partial x_{i,t-\Delta t} - \partial\nu(G^C_{t-2\Delta t})/\partial x_{i,t-2\Delta t},$$

which gives extrapolated drift, $\text{Drift}^{\text{ext}}$. Since drift is linear in the weights, $\text{Drift}^{\text{ext}}$ changes drift by $2(-\Omega_t\Delta t^2) - (-2\Omega_t\Delta t^2) = 0$ at this order, and, like lagged weights, it contains no error at t and shares no flow period with the change. Extrapolated weights treat errors that last more than one interval as true changes, so I report them for sales, where the second lag raises displacement, and not for survey wealth, where it lowers it.

Other limits of the correction. First, the correction treats as noise any transitory component of true positions, such as valuations that revert, and it leaves errors that persist beyond one interval. Second, the lagged weight contains its own error, so drift uses each unit's weight averaged over the error rather than its weight at the true position. The two coincide for the variance and the HHI, whose weights are linear in positions, and, up to granular terms, for the exponential mean when the error has the same distribution for every unit. For top shares and the Gini, whose rank weights the error smooths, they differ by a term of second order in the error that grows with the interval like the components, so comparing intervals does not detect it. Third, a correction estimated on the units observed at the earlier dates and applied to all continuing units assumes that its share of displacement is the same for the units observed at only two dates. The applications use the correction throughout (Appendices D.3, D.2.1, and D.1.2).

C.2 Choosing the Variable

The decomposition depends on the variable in which positions are recorded. With normalized positions, for a scale-invariant statistic, or with log positions, the statistic takes the same value at each date, so demography and the change among continuing units are the same; drift changes, and displacement changes by the opposite amount. Only the change from log positions is first order in time when units face idiosyncratic shocks.

Normalized positions. Let ν be scale invariant and $\tilde{x}_{it} = x_{it}/\mathrm{E}[x_{it}]$, with the mean taken over continuing units. Scale invariance gives $\partial\nu(G_t^C)/\partial x_{it} = (\partial\nu(\tilde{G}_t^C)/\partial\tilde{x}_{it})/\mathrm{E}[x_{it}]$, with $\tilde{G}_t^C$ the distribution of $\tilde{x}_{it}$. Writing $\Delta x_{it}/\mathrm{E}[x_{it}] = (\mathrm{E}[x_{i,t+\Delta t}]/\mathrm{E}[x_{it}])\Delta\tilde{x}_{it} + (\mathrm{E}[x_{i,t+\Delta t}]/\mathrm{E}[x_{it}] - 1)\tilde{x}_{it}$, the second term adds $\sum_{i\in C}\tilde{x}_{it}\partial\nu(\tilde{G}_t^C)/\partial\tilde{x}_{it} = 0$ to drift (Section 1.1), so drift in positions is drift in normalized positions times the growth of the mean, $\mathrm{E}[x_{i,t+\Delta t}]/\mathrm{E}[x_{it}]$. The gap is the growth of the mean times drift, of order Δt^2, so both decompositions have the same limit in Proposition 2; over multi-year intervals, however, the growth of the mean need not be small. The applications use normalized positions (normalized wealth and market shares), since displacement of the Gini and top shares is then exactly zero, over any interval, when no unit changes rank.

Log positions. With log positions $\ln x_{it}$, the chain rule gives each unit the weight $x_{it}\partial\nu(G_t^C)/\partial x_{it}$ on its log change, so drift changes by

$$\text{Drift}^{\ln x} - \text{Drift} = \sum_{i\in C} x_{it}\frac{\partial\nu(G_t^C)}{\partial x_{it}}\left(\ln\left(1+\frac{\Delta x_{it}}{x_{it}}\right) - \frac{\Delta x_{it}}{x_{it}}\right), \tag{C1}$$

where the superscript $\ln x$ denotes the components computed with log positions. Since $\ln(1+u) - u = -\frac{1}{2}u^2 + o(u^2)$ and $\mathrm{E}[(\Delta x_{it})^2 \mid x_{it}] = \sigma_t(x_{it})^2\Delta t + O(\Delta t^2)$, the gap in rates is the Itô term $-\frac{1}{2}\int x\partial_x \mathrm{IF}_\nu(x;G_t)\sigma_t(x)^2/x^2\,\mathrm{d}G_t(x)$, under Assumptions 1, A1, and A2(a). In continuous time, displacement in log positions weights volatility by $\frac{1}{x}\partial_x(x\partial_x\mathrm{IF}_\nu)$, which the transfer principle does not sign: for the top share, it is negative below the threshold. It is signed instead by the ex post principle of Fleurbaey and Michel (2001), which the variance of logs and generalized entropy indices with $\alpha \leq 0$ satisfy and top shares do not.

Decomposing the growth of a statistic. For a positive statistic, the decomposition of Proposition 1 adapts to the *growth* of the statistic, as a product of growth factors between consecutive values of the statistic:

$$\frac{\nu(G_{t+\Delta t})}{\nu(G_t)} = \underbrace{\frac{\nu(G_t^C) + \sum_{i\in C}\frac{\partial\nu(G_t^C)}{\partial x_{it}}\Delta x_{it}}{\nu(G_t^C)}}_{\text{Drift}} \times \underbrace{\frac{\nu(G_{t+\Delta t}^C)}{\nu(G_t^C) + \sum_{i\in C}\frac{\partial\nu(G_t^C)}{\partial x_{it}}\Delta x_{it}}}_{\text{Displacement}} \times \underbrace{\frac{\nu(G_t^C)}{\nu(G_t)}}_{\text{Exit}} \times \underbrace{\frac{\nu(G_{t+\Delta t})}{\nu(G_{t+\Delta t}^C)}}_{\text{Entry}},$$

where the sum is the additive drift of Proposition 1. To first order, each factor is one plus the additive component divided by the statistic. The multiplicative decomposition suits statistics homogeneous of degree one, $\nu(G_{\lambda x}) = \lambda\nu(G)$, such as the top p average or Atkinson-

equivalent wealth. By Euler's theorem, $\nu(G_t^C) = \sum_{i \in C} x_{it} \partial \nu(G_t^C)/\partial x_{it}$, so the drift factor is the growth factor of positions summed with their initial weights; for the top p average, the growth factor of the average position of the initial top group. Multiplying every final position by a common factor $e^{g\Delta t}$, from aggregate growth or inflation, therefore multiplies the drift factor by $e^{g\Delta t}$ and leaves the other factors unchanged, at any horizon; in the additive decomposition, it also multiplies displacement and entry by $e^{g\Delta t}$, a change of second order in time.

D Empirical Appendix

This appendix contains the details of the three applications of Section 4. Appendix D.1 treats aggregate productivity; Appendix D.2, wealth inequality; Appendix D.3, product market concentration.

D.1 Aggregate Productivity

Appendix D.1.1 describes how the growth models of Table 3 are decomposed. Appendix D.1.2 measures displacement from sales shares, with its checks; Appendix D.1.3 builds aggregate productivity growth and its components behind Table 4; Appendix D.1.4 compares the decomposition with Baqaee and Farhi (2019), Hsieh and Klenow (2009), and Aghion et al. (2019); Appendix D.1.5 checks the assumptions behind (4): measurement choices, heterogeneous markups, changes in the appeal of varieties, listed firms, and firms in place of product lines.

D.1.1 Growth Models

Table 3 reports the instantaneous rates of Proposition 2, or of its version with jumps, on each model's balanced growth path at its authors' calibration, which Chen (2026) reaches only asymptotically. Units are product lines, whoever owns them, and each model is decomposed with its own aggregator. The replication package (`code/julia`) documents each model's aggregator, calibration, and implementation.

A productivity innovation generates both drift and displacement. When a product line with a small share s_{it} raises its log productivity by J, industry productivity rises by approximately $s_{it}(e^{(\theta-1)J} - 1)/(\theta - 1)$, while Hulten's formula counts $s_{it}J$. The difference is displacement. Since $(e^{(\theta-1)J} - 1)/(\theta - 1)$ is convex in J when $\theta > 1$, dispersion in lines' steps raises growth for the same average step: displacement is the aggregate counterpart of the Oi–Hartman–Abel effect, in which a convex payoff makes volatility raise its expected value (Oi, 1961; Hartman, 1972; Abel, 1983). If each line innovates at Poisson rate λ and each innovation raises its log productivity by a random step J, then, when shares are small, instantaneous drift is $\lambda \mathrm{E}[J]$ and displacement is

$$\lambda \mathrm{E}\left[\frac{e^{(\theta-1)J} - 1}{\theta - 1} - J\right] \approx \frac{1}{2}(\theta - 1)\lambda \mathrm{E}[J^2],$$

to second order in J (Appendix B.2). With product lines as units, improvements by a line's

owner and creative destruction, which Garcia-Macia et al. (2019) separate, are both quality steps on an existing line, so they enter drift and displacement alike: displacement's share of growth depends on how dispersed the steps are, not on who makes them.

Table D1 compares the instantaneous rates of Table 3 with the components over one year, the interval of the data. In Aghion et al. (2019), Garcia-Macia et al. (2019), Peters and Walsh (2026), and Chen (2026), a line's log productivity change does not depend on its position and lines exit at random or with almost no sales, so the change among continuing lines, and with it each component, is proportional to the interval. Most other models break this independence, yet their one-year components stay close to the rates, except in Perla et al. (2021), where one-year drift counts the jump of firms that adopt within the year at their initial weight, which is larger than their weight at the adoption threshold.

Table D1: Instantaneous and One-Year Components in Growth Models

	Instantaneous rate			One year		
	Drift	Displ.	Demog.	Drift	Displ.	Demog.
Panel A: Identical technology						
Solow (1956)	2.00	0.00	0.00	2.00	0.00	0.00
Panel B: Quality-ladder models						
Aghion et al. (2019)	1.86	0.65	0.20	1.86	0.65	0.20
Garcia-Macia et al. (2019)	1.24	0.36	0.15	1.24	0.36	0.15
Peters (2020)	3.00	0.00	0.00	3.00	0.00	0.00
Peters and Walsh (2026)	3.16	0.11	−1.43	3.16	0.11	−1.43
Panel C: Idea-diffusion models						
Luttmer (2007)	0.59	1.05	0.36	0.59	1.05	0.36
Lucas and Moll (2014)	0.87	1.16	0.00	0.88	1.15	0.00
Sampson (2016)	0.00	0.00	1.55	0.00	0.00	1.56
Perla et al. (2021)	−1.20	2.00	0.00	−0.92	1.72	0.00
Chen (2026)	0.00	2.00	0.00	0.00	2.00	0.00

Notes: Displ.: displacement; Demog.: demography. Percent per year, with product lines as units. Instantaneous rate: the components of Table 3. One year: the components of Proposition 1 over one year on the balanced growth path. The two differ by at most 0.01 except in Perla et al. (2021). Simulation standard errors of the one-year components are below 0.01.

D.1.2 Measuring Displacement from Sales Shares

Sample. I use the Compustat firms incorporated in the United States, fiscal years 1984–2025, with standard filters on assets, liabilities, equity, and sales, and excluding agriculture, construction, financials, utilities, and SIC codes 5300–5399 and 9000–9999. I keep a firm from five years after its listing, dated by the start of its CRSP record (or its first year in Compustat), so that newly listed firms, which are particularly volatile, do not drive displacement. A firm's detailed product is its 4-digit SIC code, historical or, when missing, current, and its industry is the industry code most often assigned to the firms with that SIC code, so that each detailed product lies in one industry. Industry codes are mapped to the BEA industries of the input–output table; an industry code that covers several BEA industries gets the sum of their Domar weights. Industries without Compustat firms contribute nothing to displacement or the price bias of continuing firms.

Mergers. A target is a firm whose CRSP delisting record classifies it as a merger; its acquirer is the CRSP security that the record names. These records identify 1,102 mergers between firms in the sample; targets account for 2.8 to 3.8 percent of sales a year in 1997–2000 and at most 2.0 percent afterward. The target is left out of its last year in the sample, t, so that it does not exit, and the acquirer is left out of the two intervals ending at $t+1$ and $t+2$, the fiscal years in which it consolidates the target. The acquisition then enters neither displacement nor demography, at the cost of the acquirer's own growth over those two years.

Acquisitions of private or foreign firms and divestitures also move shares permanently. When they lead a firm to restate or consolidate its annual sales, its fiscal quarters no longer add up to its annual sales, so I leave the firm out of the two intervals that touch each year whose quarters are not four or do not sum to within 10 percent of its annual sales (3.6 percent of sales).

Correction for flows and noise. Transitory noise inflates relative entropy, as it inflates measured dispersion in revenue productivity (Bils et al., 2021). The annual decomposition corrects displacement with the lagged weights of (3). The correction is the drift with each firm's weight at its share one year earlier minus the drift with its weight at the start of the interval, computed on the firms present in the same industry in three consecutive years with no merger flag. Divided by displacement on the same firms, both summed over detailed products and years with the weights of displacement, it gives the ratio χ of Appendix C.1: -0.55 (-0.53 when merger targets count as exits). Displacement of every industry and year is then $1-\chi$ times its measured value; the Hulten term, a residual, absorbs the difference. Corrected relative entropy averages 0.83 percent a year, using products' sales shares within industries and normalized Domar weights across industries; divided by $\theta-1$, it gives the average industry displacement of 0.28 percent a year at $\theta=4$. The Domar weights of the covered industries sum to 1.36 on average.

Quarterly intervals. Compustat also reports quarterly sales, which allow the length of the interval to vary on the same firms. Fiscal quarters are matched to the firm-years of the sample above, with their industry and merger flags, and placed in calendar quarters; each firm's quarterly sales are divided by its average share of annual sales in that quarter of the year, since seasonal patterns differ across firms. Figure 2 uses windows of 17 consecutive calendar quarters, starting in the first quarter of each year from 1987 to 2021, keeping the firms with positive sales in every quarter, in the same industry, and with no merger flag. For an interval from quarter τ to $\tau+H$, with $s_{i\tau}$ firm i's share of the sales of its detailed product among the firms of the window, displacement within the product is

$$\text{Displacement}^{\text{lag}}_{\tau,\tau+H} = \frac{1}{\theta-1}\sum_i s_{i,\tau-4}\ln\frac{s_{i\tau}}{s_{i,\tau+H}},$$

the lagged weights of (3) at four quarters, so that the weights have the same age at every length and what remains of seasonality cancels. It is summed across products and industries as in Ta-

ble 4, annualized by four over H, and averaged over intervals and windows; the confidence interval resamples industries. Displacement is 0.32 percent a year over one quarter; with current weights it is 0.36 percent and falls with the length of the interval, as transitory movements of sales make it.

The fits in Figure 2 use displacement over H quarters with no intercept. Fitting $aH + bH^2$ gives $b = -0.0001$ (bootstrap standard error 0.0003): the quadratic term accounts for -1 percent (standard error 3) of displacement over eight quarters. Adding an intercept, for transitory movements that the lagged weights do not remove, gives a quadratic term of 10 percent (standard error 3).

Quarterly sales also check the correction. On the firms of Figure 2, annual displacement from calendar-year sales is 0.16 percent a year with current weights and 0.26 with lagged weights, against 0.26 from quarterly links over four quarters, which annual averaging does not affect. These values are below the annual estimate of Section 4.1 because these firms are present for more than four years, and firms about to exit or just entering the sample have more volatile shares. On these firms, $\chi = -0.64$, beyond the $-1/2$ of flows alone, a sign of persistent differences in growth among firms with the same share (Appendix C.1).

Weights further back raise displacement further, another sign of persistent differences in growth: on the firms present in four consecutive years, the correction is 51 percent of displacement with weights one year earlier and 69 percent with weights two years earlier (Panel B of Table D2). I keep the one-year lag as the baseline, as for the HHI, since it needs fewer dates and no assumption on the shape across lags.

Displacement and the moments of growth. Since shares move with $e^{(\theta-1)\Delta \ln A_{it}}$, the displacement of product k is exactly $\frac{1}{\theta-1}\sum_{n\geq 2}\frac{1}{n!}\kappa_n(\Delta \ln s_{it})$, where κ_n is the n-th cumulant of firms' log share growth across the continuing firms of k, weighted by $s_{it}/s_{C_k t}$. On the firms, products, years, and weights of Table 4, firms' annual log sales growth relative to their product has a standard deviation of 11 percent, a skewness of -3.5, and a kurtosis of 106, yet the variance term exceeds displacement by only 7 percent: each higher cumulant enters multiplied by a power of the standard deviation of annual growth, which is small.

D.1.3 Decomposing Aggregate Productivity Growth

Accounts. The accounts of Section 4.1 use the BEA summary use tables before redefinitions, in producer prices (U.S. Bureau of Economic Analysis, 2026b), and the business-sector growth of output, capital, hours, and labor quality of Fernald (2014). On the 66 private industries of the tables, the capital share of each industry is its gross operating surplus over its gross output and Domar weights are sales divided by GDP. Year t is growth from $t-1$ to t, with the input–output table of year t. For 1987–1996, the input–output tables come from BEA's historical tables, whose merged industries are split by their shares in 1997.

Price statistics and measured growth. Aggregate productivity growth is the measured Solow residual plus the gains from substitution and variety that deflators omit, which I infer from

sales shares under an arithmetic Laspeyres benchmark. Within a detailed product k, with C_k its continuing firms and s_{C_kt} their combined share of sales, as in Section 4.1, write P^C_{kt} for their CES price index. CES demand implies

$$\frac{p_{i,t+\Delta t}/p_{it}}{P^C_{k,t+\Delta t}/P^C_{kt}} = \left(\frac{s_{i,t+\Delta t}/s_{C_k,t+\Delta t}}{s_{it}/s_{C_kt}}\right)^{-1/(\theta-1)}.$$

The arithmetic Laspeyres relative with initial sales shares is $L_{k,t,t+\Delta t} = \sum_{i\in C_k}(s_{it}/s_{C_kt})p_{i,t+\Delta t}/p_{it}$. Its excess log inflation is therefore

$$\ln L_{k,t,t+\Delta t} - \Delta \ln P^C_{kt} = \ln \sum_{i\in C_k} \frac{s_{it}}{s_{C_kt}} \left(\frac{s_{i,t+\Delta t}/s_{C_k,t+\Delta t}}{s_{it}/s_{C_kt}}\right)^{-1/(\theta-1)} = \text{Bias}_{k,t,t+\Delta t}. \tag{D1}$$

The geometric index instead has excess inflation

$$\sum_{i\in C_k} \frac{s_{it}}{s_{C_kt}} \Delta \ln p_{it} - \Delta \ln P^C_{kt} = \text{Displacement}_{k,t,t+\Delta t}.$$

Jensen's inequality gives $\text{Bias}_{k,t,t+\Delta t} \geq \text{Displacement}_{k,t,t+\Delta t} \geq 0$. To second order in price changes, the bias is $\theta/(\theta-1)$ times displacement; the calculation uses the exact formula at each value of θ.

Entry and exit contribute separately. The CES price index of all firms satisfies $P_{kt} = P^C_{kt} s^{1/(\theta-1)}_{C_kt}$. Thus, if replacement and imputation record no net variety gain, the excess inflation of the matched arithmetic index relative to the all-firm CES index is

$$\ln L_{k,t,t+\Delta t} - \Delta \ln P_{kt} = \text{Bias}_{k,t,t+\Delta t} + \text{Demography}_{k,t,t+\Delta t},$$

with demography as in (4), the correction of Feenstra (1994). It is not rescaled by $\theta/(\theta-1)$.

Within a BEA industry, I sum the product-level biases with the products' initial shares of continuing firms' sales, under the Cobb–Douglas assumption across detailed products. Let e_{kt} denote this industry bias plus its demographic correction, b_{kt} its share of final expenditure, and ω_{kj} the share of industry j in industry k's costs. The industry's measured productivity growth is understated by $e_{kt} - \sum_j \omega_{kj} e_{jt}$, since its intermediate inputs are understated too. Since Domar weights satisfy $\lambda_{jt} = b_{jt} + \sum_k \lambda_{kt}\omega_{kj}$, the sum $\sum_k \lambda_{kt}(e_{kt} - \sum_j \omega_{kj} e_{jt})$ equals $\sum_k b_{kt} e_{kt}$: to first order across industries, the measured Solow residual understates aggregate productivity growth by the price bias and demography summed with final-expenditure shares. The Hulten term is the reconstructed total minus displacement and demography summed with Domar weights.

In the baseline sample, the arithmetic correction raises average annual growth from the measured Solow residual of 0.65 percent to 1.18 percent. Implied "missing" growth is 0.52 percentage points a year, 44 percent of aggregate productivity growth: 0.25 points from the price bias of continuing firms and 0.27 from demography. With the same annual weights and entry treatment, a geometric deflator would imply missing growth of 0.45 percentage points; arithmetic aggregation adds 0.07. The benchmark approximates the modified Laspeyres for-

mula of the PPI (U.S. Bureau of Labor Statistics, 2026b) by initial sales shares refreshed each year; official reference weights are not observed. Other deflators use geometric or Fisher formulas (U.S. Bureau of Labor Statistics, 2026a; U.S. Bureau of Economic Analysis, 2026a), so the growth correction is conditional on this benchmark.

The price bias of the deflator benchmark is corrected with the same χ, since price indices chain prices quoted at a date within the year. Where deflators are instead built from annual average prices, the price bias as measured is the relevant one: aggregate productivity growth is then 1.09 percent a year instead of 1.18, so the two values bound it.

Demography. Entry and exit in Compustat are listings and delistings, so demography comes from the Business Dynamics Statistics of the Census Bureau (2023 release), which report, by NAICS sector and firm age, the employment of all U.S. firms with paid employees and of the firms that die. As in Aghion et al. (2019), I use employment shares in place of sales shares and count a firm as entering when it reaches age five: a new firm takes time to reach the customers its products warrant, so its share at birth understates what it adds. Demography within a sector is that of Corollary 1, with continuing firms those aged at least five at t that survive to $t + \Delta t$; sector terms are summed with the Domar weights of the BEA industries of each sector. Compustat firms are older than five, so their displacement does not overlap with demography. Counting firms as entrants at birth instead gives a demography of -0.09 percent a year, aggregate productivity growth of 0.87 percent, and a Hulten term of 0.60.

Displacement between detailed products. Section 4.1 takes demand to be Cobb–Douglas across the detailed products of an industry. In the NBER-CES Manufacturing Industry Database (Becker et al., 2021), shares of shipments of 6-digit manufacturing industries rise with relative prices, with an implied elasticity of substitution of 0.55, below one, and displacement computed from their prices within BEA manufacturing industries is -0.035 percent a year from 1998 to 2013, which motivates the Cobb–Douglas approximation. Among Compustat firms, the sales-share formula between 4-digit SIC codes would add 0.11 percent a year to displacement.

D.1.4 Comparison with the Literature

Comparison with Baqaee and Farhi (2019). Baqaee and Farhi (2019) study the second-order terms that Hulten's theorem leaves out across industries. By Hulten's theorem applied to industries, the Solow residual is to first order $\sum_k \lambda_{kt} \Delta \ln \bar{A}_{kt}$, where λ_{kt} is industry k's Domar weight. What this formula leaves out is the displacement of aggregate productivity across industries: to second order, $\frac{1}{2} \sum_k \Delta\lambda_{kt} \Delta \ln \bar{A}_{kt}$, the approximation of Section 1.1 applied to industries. Industries are complements (Baqaee and Farhi, 2019), so the term is negative: slow-growing industries gain weight, as in Baumol's cost disease. It vanishes in continuous time when industries' productivity moves along paths of finite variation (Baqaee and Farhi, 2020, footnote 19), but not when industries receive shocks of their own. The Hulten term, computed as the residual, contains it.

The World KLEMS account for the United States (Jorgenson et al., 2016), which Baqaee and Farhi (2019) use for Baumol's cost disease, covers 65 industries from 1947 to 2014. Over a horizon of H years, the change is the chained Törnqvist aggregate of industries' TFP growth; drift weights each industry's log TFP change by its initial Domar weight, and displacement is the rest. Over one year, displacement across industries averages 0.00 percent a year (-0.03 with lagged weights), against 0.57 for drift. Over longer horizons, it becomes negative and grows faster than the horizon: annualized, it is -0.06 percent a year over five years, -0.07 over ten, and -0.09 over twenty (Figure D1), the second-order term of persistent differences in industries' growth (Section 2.2).

Over the 66 years of Baqaee and Farhi (2019), weights fixed in 1948 give 47.4 log points of TFP growth in these data, against 37.8 with annually chained weights, a gap of about a fifth of growth, as in their estimate. This Baumol effect is a counterfactual that builds up with the horizon, whereas displacement is a rate. Section 4.1 discusses what this persistence implies for chaining Hulten's formula (see also Appendix B.4).

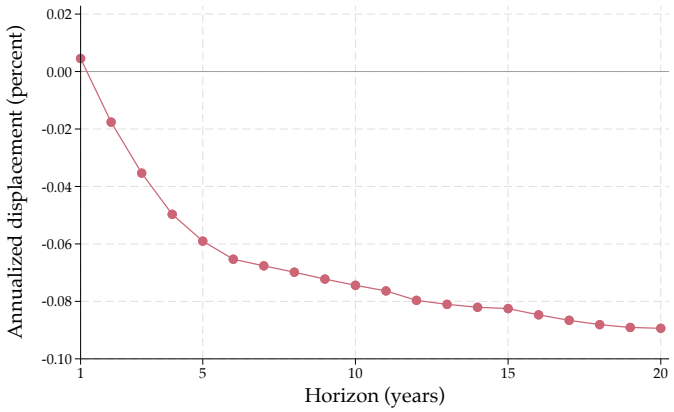

Figure D1: Displacement across Industries by Horizon

Notes: Displacement of aggregate TFP across the 65 industries of the World KLEMS account for the United States, over a horizon of H years, in percent per year, annualized by $1/H$: the chained Törnqvist aggregate of industries' TFP growth minus drift, the sum of industries' log TFP changes over the horizon weighted by their Domar weights at the start. Averages over the start years 1948 to 1994.

Comparison with Hsieh and Klenow (2009). Industry productivity $\ln \bar{A}_{kt}$ is the efficient productivity of Hsieh and Klenow (2009), which increases with the dispersion of productivity across firms. Holding the set of firms fixed, it can be written, up to the love-of-variety term of Section 4.1, which is then constant, as

$$\ln \bar{A}_{kt} = \underbrace{\mathrm{E}[\ln A_{it}]}_{\text{Average}} + \underbrace{\frac{1}{\theta-1} \ln \mathrm{E}\left[e^{(\theta-1)(\ln A_{it} - \mathrm{E}[\ln A_{it}])}\right]}_{\text{Dispersion}},$$

where E is the unweighted average across the firms of industry k. The second term measures the dispersion of log productivity around its average: it is zero when all firms have the same productivity and positive otherwise; when dispersion is small, it is approximately $(\theta-1)/2$ times the variance of log productivity. Their measure of misallocation is a separate term, the

dispersion of revenue productivity, which is zero here because firms charge the same markup (Appendix D.1.5).

Taking changes relates this split to drift and displacement. The change in industry productivity is

$$\Delta \ln \bar{A}_{kt} = \underbrace{\underbrace{\mathrm{E}[\Delta \ln A_{it}]}_{\substack{\text{Drift (average part)} \\ = \text{Change in average}}} + \underbrace{\sum_{i \in k} s_{it} \left(\Delta \ln A_{it} - \mathrm{E}[\Delta \ln A_{it}] \right)}_{\text{Drift (covariance part)}} + \underbrace{\frac{1}{\theta - 1} \ln \sum_{i \in k} s_{it} e^{(\theta-1)\Delta \ln A_{it}} - \sum_{i \in k} s_{it} \Delta \ln A_{it}}_{\text{Displacement}}}_{\text{Change in dispersion}},$$

where s_{it} is firm i's share of the sales of the industry. The two parts of drift are those of (1). The dispersion term therefore rises for two reasons: firms with large sales improve faster than the average firm, which is part of drift, or firms' changes are dispersed, which is displacement.

Comparison with Aghion et al. (2019). Aghion et al. (2019) measure a related term on plants: 0.54 percentage points a year over 1983–2013 (0.23 with entry at birth). It differs from demography here because they measure the part of entry's contribution that measured growth leaves out, with final-expenditure weights, whereas demography is the whole contribution of entry and exit, with Domar weights; and because a new plant of an existing firm is entry for them but growth of a continuing firm here.

D.1.5 Robustness

Measurement choices. Panel A of Table D2 changes one choice of Table 4 at a time, and Panel B the date of the weights on one set of firms. Displacement and demography stay large in every row of Panel A: displacement ranges from 0.16 to 0.38 percent a year and demography from 0.20 to 0.76. Domestic sales barely change displacement; 6-digit NAICS codes lower it to 0.33 percent a year, since displacement between products is set to zero; leaving out catch-all SIC codes gives 0.37; extrapolating the correction to no lag gives 0.31. Counting establishments rather than firms raises demography to 0.76 percent a year, since a new plant of an existing firm then counts as entry.

Given the observed shares, displacement and demography scale with $1/(\theta - 1)$, and the measured Solow residual does not depend on θ; the arithmetic price bias is recomputed from its exact formula. With $\theta = 8$, the value of Baqaee and Farhi (2020), displacement and demography are 0.16 and 0.20 percent a year, out of aggregate productivity growth of 0.86; together they account for 42 percent of growth, and displacement for 20 percent of the slowdown after 1999. With the elasticity of each 2-digit sector implied by the markups of Hall (2018), as in Aghion et al. (2019), displacement and demography account for 62 percent of growth.

Heterogeneous markups. When markups differ across the firms of a detailed product, sales shares move with productivity relative to the markup, so displacement and demography measured from sales also record changes in markups, and aggregate productivity gains a term for changes in allocative efficiency (Hsieh and Klenow, 2009; Baqaee and Farhi, 2020). With equal

Table D2: Decomposing Aggregate Productivity Growth: Robustness

	Total	Drift	Displacement	Demography
Panel A: Baseline and other choices				
Baseline	1.18	0.33	0.37	0.48
Domestic sales	1.18	0.32	0.38	0.48
4-digit NAICS	1.16	0.31	0.38	0.48
6-digit NAICS	1.13	0.33	0.33	0.48
Without catch-all SIC codes	1.17	0.32	0.37	0.48
Sector-specific elasticities	1.07	0.40	0.35	0.32
Entry/exit at establishment level	1.34	0.20	0.37	0.76
Higher elasticity $\theta = 8$	0.86	0.50	0.16	0.20
Weights extrapolated to no lag	1.13	0.35	0.31	0.48
Panel B: Date of the weights, on one set of firms				
Weights at t			0.19	
Weights at $t-1$			0.29	
Weights at $t-2$			0.33	

Notes: Annual growth in percent, averaged over all years. Baseline: the All years row of Table 4. Each other row of Panel A changes one choice. Domestic sales: each firm's sales times the domestic share of its geographic segments. 4- and 6-digit NAICS: the detailed product is the NAICS code. Without catch-all SIC codes: firms whose SIC code ends in 0 or 9 are left out, so the other products of their industry stand in for them. Higher elasticity $\theta = 8$ and sector-specific elasticities (from the markups of Hall, 2018): the elasticity of substitution between firms within a detailed product, $\theta = 4$ in every other row. Entry/exit at establishment level: demography from establishments in the Business Dynamics Statistics. Weights extrapolated to no lag: the correction of twice lagged displacement at one year minus that at two (Appendix C.1). Panel B computes displacement directly on the firms present in the same industry in four consecutive years, over the final annual interval, with each firm's weight at its share at t (uncorrected), $t-1$, or $t-2$; it reports only displacement, since sales do not identify the level of drift.

markups, as in the baseline, the dispersion of revenue productivity, the measure of misallocation of Hsieh and Klenow (2009), is zero. The decomposition of Baqaee and Farhi (2020) applies to a distortion-adjusted Solow residual, which weights factor growth by cost-based shares. Over 1998–2013, on their firm file and with their user-cost markups, displacement is 0.78 percent a year and demography 0.43, close to their values without markups. The Hulten term, a residual, falls to -0.10, by about as much as allocative efficiency, 0.40.

Changes in appeal. Productivity in Section 4.1 includes the appeal of a firm's variety, which accounts for most of the variance of firms' sales in scanner data (Hottman et al., 2016). The mapping from sales shares to effective productivity uses the normalization of Redding and Weinstein (2020): changes in tastes average zero across firms and are uncorrelated with firm size. Changes in input prices, financial frictions, and adjustment costs can also move sales shares and are unobserved.

Corollary 1 is an identity, so it still holds when the appeal of firms' varieties changes between two dates: effective productivity simply includes appeal. What the corollary measures then depends on why appeal changes. If a variety improves in quality, which has units (the ounces in a box of cereal or the speed of a processor), it gives the gain to buyers, and displacement is a gain; Redding and Weinstein (2020) adopt this interpretation. If tastes shift, choices do not reveal whether a buyer with today's tastes is better off than a buyer with yesterday's, so the comparison has to hold tastes fixed (Fisher and Shell, 1968; Baqaee and Burstein, 2023).

Buyers combine firms' output into the industry's output, $Y_{kt} = (\sum_{i \in k} (\varphi_{it} y_{it})^{(\theta-1)/\theta})^{\theta/(\theta-1)}$,

where φ_{it} is the appeal of firm i's variety. Firm i produces Z_{it} units of output per unit of the input bundle, its efficiency, so its effective productivity is $A_{it} = \varphi_{it} Z_{it}$, and its relative price moves only with its efficiency: $\Delta \ln p_{it} = -\Delta \ln Z_{it}$ up to a term common to the industry. Technology growth at initial appeal is $\ln \hat{A}_{k,t+\Delta t} - \ln \bar{A}_{kt}$, where $\ln \hat{A}_{k,t+\Delta t} \equiv \frac{1}{\theta-1} \ln \sum_{i \in k} e^{(\theta-1)(\ln \varphi_{it} + \ln Z_{i,t+\Delta t})}$ is industry productivity at $t + \Delta t$ if the firms present at that date had their initial appeal; a change in tastes alone leaves this growth at zero. Applying Corollary 1 along these counterfactual positions gives its decomposition.

Corollary D1 (Decomposition of Industry Productivity at Initial Appeal)**.** *Productivity growth at initial appeal in each industry k can be decomposed as follows:*

$$\ln \hat{A}_{k,t+\Delta t} - \ln \bar{A}_{kt} = \underbrace{\sum_{i \in C_k} \frac{s_{it}}{s_{C_k t}} \Delta \ln Z_{it}}_{\text{Drift}} + \underbrace{\frac{1}{\theta - 1} \sum_{i \in C_k} \frac{s_{it}}{s_{C_k t}} \ln \frac{s_{it}/s_{C_k t}}{\hat{s}_{i,t+\Delta t}/\hat{s}_{C_k,t+\Delta t}}}_{\text{Displacement}} + \underbrace{\frac{1}{\theta - 1} \ln \frac{s_{C_k t}}{\hat{s}_{C_k,t+\Delta t}}}_{\text{Demography}},$$

where C_k is the set of firms of industry k present at both dates, s_{it} is firm i's share of the sales of industry k, $s_{C_k t} \equiv \sum_{i \in C_k} s_{it}$, $\hat{s}_{i,t+\Delta t} \equiv (\varphi_{it} Z_{i,t+\Delta t})^{\theta-1} / \sum_{j \in k} (\varphi_{jt} Z_{j,t+\Delta t})^{\theta-1}$ is firm i's share at $t + \Delta t$ at initial appeal, and $\hat{s}_{C_k,t+\Delta t} \equiv \sum_{i \in C_k} \hat{s}_{i,t+\Delta t}$.

Since $\hat{s}_{i,t+\Delta t}/\hat{s}_{C_k,t+\Delta t}$ is proportional to $(s_{it}/s_{C_k t}) e^{(\theta-1)\Delta \ln Z_{it}}$, displacement is equivalently $\frac{1}{\theta-1} \ln \sum_{i \in C_k} (s_{it}/s_{C_k t}) e^{(\theta-1)\Delta \ln Z_{it}} - \sum_{i \in C_k} (s_{it}/s_{C_k t}) \Delta \ln Z_{it}$. It is non-negative, as a relative entropy, and moves only because efficiency changes. Since relative prices move only with efficiency, it can be written with prices, $\sum_{i \in C_k} (s_{it}/s_{C_k t}) \Delta \ln p_{it} + \frac{1}{\theta-1} \ln \sum_{i \in C_k} (s_{it}/s_{C_k t}) (p_{i,t+\Delta t}/p_{it})^{1-\theta}$: the log of a geometric Laspeyres index of continuing firms' prices minus the log of a CES index at initial appeal.

Let Var_s and Cov_s denote the variance and covariance across continuing firms with weights $s_{it}/s_{C_k t}$. To second order in the changes, the displacement of Corollary 1 is

$$\frac{\theta - 1}{2} \mathrm{Var}_s(\Delta \ln Z_{it}) + \frac{\theta - 1}{2} \mathrm{Var}_s(\Delta \ln \varphi_{it}) + (\theta - 1) \mathrm{Cov}_s(\Delta \ln Z_{it}, \Delta \ln \varphi_{it}).$$

In Corollary D1, only the first of these terms remains. The second drops out because, at given tastes, buyers gain nothing from switching to the varieties they now like more; the third, because the weights keep appeal at its initial level. Holding appeal at its final value instead, as Baqaee and Burstein (2023) prefer for welfare, gives the same displacement to second order, while drift adds the covariance. Demography is the same as in (4) when entrants would have taken their observed share at initial appeal; if they sell only because tastes moved toward them, entry contributes nothing.

Displacement in Corollary D1 needs only initial shares, price changes, and θ, but Compustat reports no prices, so I cannot compute it for the firms of Table 4. In consumer goods, where prices are observed, Hottman et al. (2016) find that changes in appeal account for about 90 percent of the variance of firm growth. Table 4 reports Corollary 1, productivity including appeal.

It is the right measure to the extent that changes in appeal reflect quality; its total rests on the same interpretation, since inferring from sales shares the gains that deflators omit treats every change in a firm's share as a change in its effective price. To the extent that changes in appeal reflect tastes, its displacement differs from that of efficiency by the last two terms above.

Listed firms. Compustat covers listed firms, whereas Corollary 1 concerns all firms. Relative entropy among all firms is the sales-weighted average of that among listed and unlisted firms, plus the non-negative relative entropy between the two groups' shares. The listed-firm estimate therefore understates the full-population relative entropy whenever that among unlisted firms is at least as large.

Firms and product lines. The decomposition remains exact with firms as units. If a firm's lines share the elasticity θ, define firm i's effective productivity by $A_{it}^{\theta-1} \equiv \sum_{j\in i} A_{jt}^{\theta-1}$, summing over its lines j. Its sales remain proportional to $A_{it}^{\theta-1}$ and industry productivity is unchanged, so Corollary 1 holds for firms. Growth is therefore the same, but its split differs. If lines do not move between firms, line displacement equals firm displacement plus the sales-weighted displacement within firms, so firm-level drift exceeds line-level drift. A line that changes hands instead moves firm-level drift and displacement: a firm's productivity growth includes the lines it gains and loses, so firm-level drift is not the expected improvement of existing products. The same choice of units moves demography: young firms' improvements over their first five years and new firms taking over existing lines count as entry in the data, but as steps on continuing lines in the quality-ladder models of Table 3.

D.2 Wealth Inequality

D.2.1 Data and Method

Data. I use two European household wealth surveys with panel components. The Household Finance and Consumption Survey (HFCS; Household Finance and Consumption Network, 2026) covers four consecutive wave pairs (waves 1 through 5, ∼2010–2023) for Germany, Spain, and Italy, and three (waves 2 through 5, ∼2014–2023) for France.[26] The UK Wealth and Assets Survey (WAS; Office for National Statistics, Social Survey Division, 2026) covers five consecutive round pairs (wave 3 through round 8, ∼2010–2022) for Great Britain. All results are annualized by the number of years between the two survey years and averaged across wave pairs within countries; the components are then averaged across countries with equal weights.

Wealth is net wealth in the HFCS countries (variable `dn3001`). For the UK, wealth is ONS total household wealth net of pension wealth (occupational defined-benefit, defined-contribution, and personal pensions); HFCS net wealth likewise excludes occupational and public pension entitlements. In each wave, I normalize each country's wealth by its weighted mean, taken

[26] Among HFCS country-wave pairs with panel data, I retain those with at least 1,500 panel households. HFCS net wealth is multiply imputed in five implicates, and all results average across implicates.

over the full cross-section for the total change and over continuing households for drift and displacement (Table 1).

Method. I decompose the Gini coefficient, which weights every percentile, and the top 10% share, whose displacement comes only from households crossing the 90th percentile (Section 4.2); Appendix D.2.3 also decomposes the negative log of Atkinson-equivalent normalized wealth. In a finite weighted sample, $w(R_{it})$ denotes the average of w over the percentile interval occupied by household i. Demography is the residual between the total change, computed from the full cross-section in each wave, and the change among continuing households, which I reweight to represent all households that continue between the two waves (Appendix D.2.1).

Survey wealth is measured with error, which biases drift down and displacement up (Section 3). I therefore evaluate the drift weights at wealth one wave earlier for households observed in three consecutive waves, and move the estimated share of displacement due to noise into drift for all continuing households (Appendix D.2.1). The correction is large (Table D3). The interval-length identity (Appendix C.1) measures the same correction: on these households, noise accounts for about 66 percent of the Gini displacement over one wave pair, averaged across the five countries. Under the assumptions of the correction, most of the drift and displacement measured without it reflects transitory variation in measured wealth, from reporting error or from temporary changes in valuations.

Continuing households and demography in survey panels. Drift and displacement should cover all households that continue from one wave to the next, and demography only the reorganization of households: exits through death or dissolution, and the formation of new households. A large part of each wave is not re-interviewed in the next, mostly by design (rotating panel components and refresh samples) or for nonresponse, and the public files do not distinguish these households from those that dissolved. With their cross-sectional survey weights, re-interviewed households would represent only the part of the population that the survey follows, and demography would mostly reflect the panel design.

I therefore reweight the re-interviewed households to represent the population at t net of identified exits. A household whose members form several households at $t + \Delta t$ (a split) exits, and each resulting household enters. A household dissolves through death with the probability that all its adult members die before the next wave, from Eurostat period life tables by country, sex, and age of the reference person; with two or more adults, it is the probability that both the reference person and a partner of the same age and the other sex die. The survey weights of re-interviewed households are then raked to the margins of the wave-t cross-section, net of these exits: wealth bins (cut at each decile, at the 95th and 99th percentiles, and at the 99.5th and 99.9th where each top bin keeps at least 20 re-interviewed households), the age of the reference person, and, in the HFCS, household size. Raking treats attrition and rotation as ignorable given these characteristics. The same weights are used at both dates, so the decomposition remains an exact identity. Households observed in three or four consecutive waves are raked in the same way to the cross-section of their first wave in the HFCS; in the

WAS, they keep their two-wave weights.

Demography then has two parts. The exit part, $\nu(G_t^C) - \nu(G_t)$, reflects only identified exits. The entry part, $\nu(G_{t+\Delta t}) - \nu(G_{t+\Delta t}^C)$, reflects the entry of new households, but also differences between re-interviewed households and the rest of the later cross-section that are not demographic: attrition related to wealth changes after t, sample design, or panel conditioning. These differences are not negligible, even among households with a middle-aged reference person, and cannot be separated from true entry. I leave them in demography rather than also calibrating continuing households to the later cross-section, which would move them into drift and displacement.

Correction for measurement error. The correction, $\text{Drift}^{\text{lag}} - \text{Drift}$ (Appendix C.1), is measured on households observed in three consecutive waves; for the UK, the WAS panels are linked back one wave or round (wave 2 to round 7). For the Gini, which is twice the integral of the top p shares over p minus one (Appendix E), I correct the curve of top shares in levels. For each top p share and each country, I estimate the ratio $\chi(p)$ of Appendix C.1 as the level correction summed over wave triples divided by the level displacement summed over the same triples, each averaged across implicates, and interpolate it linearly between thresholds. I constrain this ratio to be at most one, so that corrected displacement keeps the sign that it has at any horizon for an ordered weighted average (Section 4.2). It has no lower bound: a negative ratio would mean that a bias toward momentum, such as that of flows, outweighs noise (Section 3).

For each wave pair, including the first, the correction moves a share $\chi(p)$ of the displacement of the top p share into its drift. The corrected Gini drift and displacement are twice the integrals over $p \in [0, 1]$ of the corrected components of the top shares, computed exactly; the household at the boundary of each top group contributes the fraction of its survey weight that gives the group a population share of exactly p. For the log of Atkinson-equivalent normalized wealth, I move into drift, at every value of ϵ and floor, the country-level ratio of the summed level correction to summed displacement, bounded above by one. All corrections leave the change among continuing households and demography unchanged.

Robustness. Table D3 reports the cross-country average of the Gini decomposition of Table 5, changing one choice at a time in Panel A and the date of the weights in Panel B. Across the rows of Panel A, average drift ranges from -1.1 to -1.0 points per year and displacement from 0.6 to 0.7; every row keeps the signs of both. In the baseline, evaluating the weights one wave earlier shrinks Gini drift by almost two thirds and Gini displacement by about four fifths. Panel B takes the weights back a second wave, which shows whether changes over consecutive intervals are correlated and whether errors last beyond one wave, on households observed in four consecutive waves over their final wave pair, where the change in the Gini does not depend on the weights. The first wave back shrinks average Gini drift and displacement by more than half, close to the baseline: drift goes from -2.5 to -1.0 points a year and displacement from 2.3 to 0.7. The second wave back changes them little.

Table D3: Decomposing the Change in the Gini of Wealth: Robustness

	Total	Drift	Displacement	Demography
Panel A: Baseline and other choices				
Baseline	0.1	−1.1	0.6	0.6
Published weights, not raked	0.1	−1.0	0.6	0.5
Excluding the last wave pair	0.2	−1.0	0.6	0.6
Weights at $t-1$, net of their error	0.1	−1.1	0.7	0.6
Panel B: Date of the weights, on one set of households				
Weights at t	−0.3	−2.5	2.3	
Weights at $t-1$	−0.3	−1.0	0.7	
Weights at $t-2$	−0.3	−1.0	0.7	

Notes: Gini of normalized wealth, in points per year, with countries weighted equally. Each row of Panel A is the Average row of Table 5 when one choice changes. Published weights: re-interviewed households keep their cross-sectional survey weights, without removing splits and deaths or raking (Appendix D.2.1). Last wave pair: HFCS waves 4–5 and WAS rounds 7–8. Weights at $t-1$, net of their error: the baseline, with the error that the lagged weights themselves contain (Appendix C.1) removed by simulation and extrapolation (Cook and Stefanski, 1994). Panel B computes drift and displacement directly, with no scaling, on households observed in four consecutive waves, over the last wave pair of each four waves, with the drift weights at wealth at t (uncorrected), $t-1$, or $t-2$. Total: the change in the Gini among these households; demography does not apply. The levels of Panel B differ from the baseline because these households are observed for longer. France enters only with waves 4–5.

D.2.2 Growth Profiles

Figure 5 uses households observed in three consecutive waves: HFCS waves 1–3, 2–4, and 3–5 for Germany, Italy, and Spain, 2–4 and 3–5 for France, and WAS waves and rounds for the UK (W2–W4, W3–W5, W4–R6, R5–R7, R6–R8), with the weights of Appendix D.2.1, averaged across implicates and wave triples. Panel (a) ranks households by wealth in the previous wave, which avoids ranking them on the measurement error in wealth at the start of the interval. Ranking households on wealth at the start of the interval instead produces a steeper gradient, with losses across countries of 4.6 to 8.9 percent per year relative to mean wealth at the 99th percentile, instead of 0.8 to 2.8: a household measured as rich in one wave is partly one whose wealth was overstated. Panel (b) reports volatility net of measurement noise: its square is the variance of growth plus twice its covariance with growth over the previous interval, which removes the transitory error in wealth at the start of the interval.

The wealth of households in the top 1 percent in the previous wave falls in real terms by 1.2 percent a year on average across countries, while mean wealth grows by 1.9 percent. The richest tenth hold about 43 percent of wealth at the start of the interval and lose about 1.9 percent a year relative to the mean. Holding the previous wave's rank weights fixed, suppose their observed total loss of normalized wealth were proportional to their starting wealth and the other households gained the same amount in proportion to their positive starting wealth. This counterfactual produces negative Gini drift of 0.5 points a year on average across countries, almost half the observed drift on the same three-wave sample.

Saving and returns. Saving and returns push drift in opposite directions. Saving rates out of income are about flat across the wealth distribution net of capital gains (Fagereng et al., 2019), so saving relative to wealth falls with wealth, which pushes drift down; returns rise with wealth (Fagereng et al., 2020; Bach et al., 2020), which pushes it up. In Sweden, higher

returns add 4.4 percentage points a year to the growth of the share held by those initially in the top 0.01 percent, but saving out of labor income, net of consumption, subtracts 8.0 (Bach et al., 2018): saving more than offsets the return advantage. Rising top wealth shares coexist with negative drift in Sweden and positive drift in the U.S. Forbes 400. In Sweden, the turnover term of Bach et al. (2018) and Hammar et al. (2026), which corresponds to displacement, more than offsets the relative decline of households initially at the top. In the Forbes 400 over 2010–2023, those already at the top grow faster than aggregate wealth (Gomez, 2023, with the Forbes lists through 2023).

D.2.3 Relation to Genicot and Ray (2023)

The upward mobility of Genicot and Ray (2023) over an interval is the annualized log growth of Atkinson-equivalent income, $\xi(G_t) = \mathrm{E}[x_{it}^{1-\epsilon}]^{1/(1-\epsilon)}$, where their parameter is $\epsilon - 1 \geq 0$. Relative upward mobility is the log growth of $\xi(G_t)/\mathrm{E}[x_{it}]$, a decreasing transformation of $\mathrm{GE}(1-\epsilon)$. The formulas below assume positive wealth.

Proposition D1 (Decomposition of the Growth of Atkinson-Equivalent Wealth)**.** *If positions follow the diffusion of Assumption 1, without entry or exit,*

$$\frac{\mathrm{d}\ln \xi(G_t)}{\mathrm{d}t} = \underbrace{\frac{\mathrm{E}[x_{it}^{1-\epsilon}\mu_t(x_{it})/x_{it}]}{\mathrm{E}[x_{it}^{1-\epsilon}]}}_{\text{Drift}} + \underbrace{-\frac{\mathrm{E}\left[x_{it}^{1-\epsilon}\frac{\epsilon\sigma_t(x_{it})^2}{2x_{it}^2}\right]}{\mathrm{E}[x_{it}^{1-\epsilon}]}}_{\text{Displacement}}, \tag{D2}$$

$$\frac{\mathrm{d}\ln(\xi(G_t)/\mathrm{E}[x_{it}])}{\mathrm{d}t} = \underbrace{\frac{\mathrm{E}[x_{it}^{1-\epsilon}\mu_t(x_{it})/x_{it}]}{\mathrm{E}[x_{it}^{1-\epsilon}]} - \frac{\mathrm{E}[\mu_t(x_{it})]}{\mathrm{E}[x_{it}]}}_{\text{Drift}} + \underbrace{-\frac{\mathrm{E}\left[x_{it}^{1-\epsilon}\frac{\epsilon\sigma_t(x_{it})^2}{2x_{it}^2}\right]}{\mathrm{E}[x_{it}^{1-\epsilon}]}}_{\text{Displacement}}. \tag{D3}$$

Proof of Proposition D1. The influence function of $\ln\xi$ is $(x^{1-\epsilon}/\mathrm{E}[x_{it}^{1-\epsilon}] - 1)/(1-\epsilon)$, with first derivative $x^{-\epsilon}/\mathrm{E}[x_{it}^{1-\epsilon}]$ and second derivative $-\epsilon x^{-\epsilon-1}/\mathrm{E}[x_{it}^{1-\epsilon}]$. Substituting them into Proposition 2 gives (D2). Mean wealth is linear in the distribution, so it has no displacement, and its log growth rate is $\mathrm{E}[\mu_t(x_{it})]/\mathrm{E}[x_{it}]$. Subtracting gives (D3). □

Drift in (D2) is the absolute mobility kernel of equation (5) of Genicot and Ray (2023), and drift in (D3) their relative kernel of equation (6): a growth rate averaged with welfare weights $x^{1-\epsilon}/\mathrm{E}[x_{it}^{1-\epsilon}]$, which favor the poor. Their measure updates these weights continuously along each path; drift holds them at the start of each instant, which separates expected growth from the risk that displacement records. With utility $u(x) = x^{1-\epsilon}/(1-\epsilon)$, $\xi(G_t)$ is the certainty equivalent of average utility, and by the Arrow–Pratt approximation (Arrow, 1965; Pratt, 1964) a household at x would pay a fraction $\frac{1}{2}\epsilon\sigma_t(x)^2/x^2$ of its wealth per unit of time to remove the risk it bears. Displacement is minus this relative risk premium, averaged with the same welfare weights as drift: the welfare cost of risk, to second order in households' changes.

Over an interval, I decompose $\ln\xi$ by taking the logs of the factors of its multiplicative decomposition (Appendix C.2). Since ξ is homogeneous of degree one in wealth, Euler's theorem

gives, with expectations over continuing households,

$$\text{Drift}^{\ln\xi} = \ln \mathrm{E}\left[\frac{x_{it}^{1-\epsilon}}{\mathrm{E}[x_{it}^{1-\epsilon}]}\frac{x_{i,t+\Delta t}}{x_{it}}\right], \qquad \text{Displacement}^{\ln\xi} = \ln \frac{\mathrm{E}\left[\frac{x_{it}^{1-\epsilon}}{\mathrm{E}[x_{it}^{1-\epsilon}]}\left(\frac{x_{i,t+\Delta t}}{x_{it}}\right)^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}}}{\mathrm{E}\left[\frac{x_{it}^{1-\epsilon}}{\mathrm{E}[x_{it}^{1-\epsilon}]}\frac{x_{i,t+\Delta t}}{x_{it}}\right]}:$$

the log of the welfare-weighted mean growth of wealth, and the log of the welfare-weighted certainty-equivalent growth relative to it. In normalized wealth, drift is lower by the log growth of mean wealth and displacement is the same.

Implementation and results. I use the relative version of Genicot and Ray (2023), with their benchmark parameter, 0.5, which is $\epsilon = 3/2$ here, and report the negative of the relative log statistic and of each component, so that the statistic rises with inequality and its drift is minus relative upward mobility. The reported statistic is $-\ln \mathrm{E}[\tilde{x}_{it}^{-1/2}]^{-2}$, an increasing transformation of GE($-1/2$) (Appendix E.4). It requires positive wealth and puts more weight on households with little wealth, whose large growth rates can dominate it, so I floor wealth at 10 percent of its weighted mean, then renormalize it to mean one, separately at each date in the cross-section and among continuing households.

Table D4 reports the decomposition, averaged across countries. The change in the statistic averages 0.4 log points a year. Drift, minus relative upward mobility, averages -5.8 log points a year: growth is pro-poor on average. Displacement raises the statistic by 4.4 log points, offsetting 77 percent of drift, and demography adds another 1.8. Among continuing households, drift and displacement lower the statistic on average, so its average rise comes from demography.

Without the correction for measurement error, drift would be -15.1 log points. The size of drift and displacement depends on how the bottom of the distribution is treated, but not the signs of their cross-country averages: across $\epsilon \in \{1, 3/2, 2, 3\}$ and floors of 1, 5, 10, and 20 percent of mean wealth, drift lowers the reported statistic by at least 2.3 log points a year on average across countries, and displacement and demography more than offset it in every specification. Within countries, drift and displacement are strongly negatively correlated across wave pairs (-0.88 for GE(0), the case $\epsilon = 1$): a few households with little wealth and large growth rates dominate drift in a given pair, so the split is informative only on average across wave pairs.

D.2.4 Comparison with Models of the Wealth Distribution

Table 6 decomposes wealth inequality in eight published models of the wealth distribution, each at its authors' calibration where they give one. Blanchard (1985) gives no calibration: I set the death rate to $1/40$, the capital share to 0.36, depreciation to 0.08, and the discount factor to 0.96, with the interest rate clearing the capital market. The replication package describes how each model is implemented.

For each model, I simulate households from its stationary distribution over three years,

Table D4: Decomposing the Change in Negative Log Atkinson-Equivalent Wealth, Average across Countries

Total	Drift	Displacement	Demography
0.4	−5.8	4.4	1.8
(0.2)	(1.8)	(1.8)	(0.2)

Notes: $-\ln \mathrm{E}[\tilde{x}_{it}^{-1/2}]^{-2}$, the negative log of Atkinson-equivalent normalized wealth with inequality aversion $\epsilon = 3/2$, which corresponds to the benchmark of Genicot and Ray (2023); drift is minus their relative upward mobility, and displacement is the welfare cost of risk. Equally weighted average across the five countries of annual averages over wave pairs, in log points per year; wealth is floored at 10 percent of its weighted mean and then renormalized to mean one. Samples, correction for measurement error, averaging, and standard errors as in Table 5; each draw also recomputes the bottom-coding.

about the interval between survey waves, and decompose the change in the Gini and the top 10% share as in Corollary 2. Households that die count as exiters and the households that replace them, heirs included, as entrants, as in the survey panels (Section 3). The models are stationary, so their three components sum to zero; I scale them to do so exactly, which removes simulation error. As in the data, drift is corrected with lagged weights, three years earlier (Appendix D.2.1). The simulated panels have no measurement error, but lagged weights also move persistent differences in expected growth from drift to displacement (Section 3), so model and data must be measured the same way.

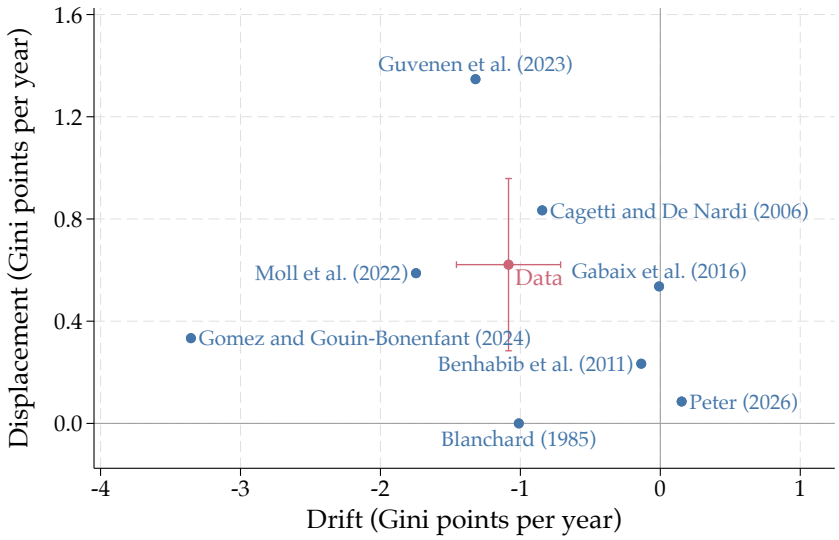

Figure D2: Drift and Displacement in Wealth Models and Data

Notes: Gini points per year. Model components are those of Table 6: changes over three years, annualized, with households that die counted as exiters and their replacements, heirs included, as entrants; drift uses lagged weights. The components are adjusted to sum to zero as in the table. The red point is the equally weighted country average of Table 5; horizontal and vertical bars are marginal 95 percent confidence intervals, using its standard errors. The models use their published calibrations and need not match the same wealth distribution.

D.3 Product Market Concentration

Demography. The entry term splits into the dilution of continuing brands and the entrants' own concentration; the exit term mirrors it. With $\mathrm{HHI}^{C}_{t+\Delta t}$ the continuing brands' HHI and

$s_{C,t+\Delta t}$ their combined share, as in Corollary 3,

$$\text{Entry} = \text{HHI}_{t+\Delta t} - \text{HHI}^C_{t+\Delta t} = \underbrace{-(1 - s^2_{C,t+\Delta t})\text{HHI}^C_{t+\Delta t}}_{\text{Dilution}} + \underbrace{\sum_{i \notin C} s^2_{i,t+\Delta t}}_{\text{Entrants}},$$

and the exit term is $(1 - s^2_{Ct})\text{HHI}^C_t - \sum_{i \notin C} s^2_{it}$, where the sums over $i \notin C$ run over the brands that enter, at $t + \Delta t$, and over those that exit, at t. Entrants with a small combined share $1 - s_{C,t+\Delta t}$ lower the index by about $2\text{HHI}^C_{t+\Delta t}(1 - s_{C,t+\Delta t})$, while their own concentration adds a term of second order in their shares. Unlike the mass term of Section 1.1, dilution depends on the entrants only through their combined share, not their number.

Granularity in shares. With few brands, a brand's own shocks move its module's sales. If the sales of each brand follow $\mathrm{d}x_{it} = \mu_{it}\,\mathrm{d}t + \sigma_{it}\,\mathrm{d}W_{it}$, with independent Brownian motions, Itô's lemma adds to the expected rate of change of a brand's share two terms that are absent with a continuum, $s_{it}(\sum_j s^2_{jt}\sigma^2_{jt}/x^2_{jt} - s_{it}\sigma^2_{it}/x^2_{it})$. They make drift negative when all brands have the same expected growth and volatility and shares are unequal. In sales, these terms are part of the granular term of Appendix B.3 and enter displacement; decomposing in shares moves them to drift. Displacement in shares is then approximately $\sum_j s^2_{jt}\sigma^2_{jt}/x^2_{jt}$, the square of the granular volatility of Gabaix (2011), when every brand's share is small. When the variance of a brand's sales growth is inversely proportional to its share, $\sigma^2_{it}/x^2_{it} = \bar{\sigma}^2/s_{it}$, the two terms cancel, and displacement is $\bar{\sigma}^2(1 - \text{HHI}_t)$, with $\text{HHI}_t = \sum_i s^2_{it}$ the HHI on the 0–1 scale.

Data and results. The sample, the robustness checks, the measurement of the profiles, and the calibration of a model of competition for sales use the NielsenIQ Retail Scanner data and are omitted from this version pending NielsenIQ review.

E The Decomposition for Common Statistics

For each statistic, this appendix gives drift and displacement over an interval where Table 1 does not; demography over an interval is entry plus exit, each measured relative to continuing units (Section 1.1), and needs no derivation. It then gives the influence function, from which Proposition 2 gives the three components in continuous time (Table 2 collects drift and displacement). Finally, it relates earlier accounting frameworks to the components. Table E1 lists these frameworks, each for the change in one distributional statistic; the paragraph on related literature in the subsection on each statistic gives the details.

In continuous time, positions follow the diffusion of Assumption 1, with expected change $\mu_t(x)$ and volatility $\sigma_t(x)$. Inequality indices are written in normalized positions $\tilde{x} = x/\mathrm{E}[x_{it}]$, whose expected change is $\tilde{\mu}_t(\tilde{x}) \equiv (\mu_t(x) - \tilde{x}\mathrm{E}[\mu_t(x_{it})])/\mathrm{E}[x_{it}]$ and whose volatility is $\tilde{\sigma}_t(\tilde{x}) \equiv \sigma_t(x)/\mathrm{E}[x_{it}]$. For population-invariant statistics, normalize the population to $N_t = 1$, which is without loss of generality.

Demography is written with the rates and averages of entry and exit. Let $n^+_t \equiv \Psi^+_t(\mathbb{R})$

and $n_t^- \equiv \Psi_t^-(\mathbb{R})$ be the masses of units that enter and exit per unit of time, so that $\mathrm{d}N_t/\,\mathrm{d}t = n_t^+ - n_t^-$; with $N_t = 1$, they are the entry and exit rates. Let $\mathrm{E}_t^+[f(x)] \equiv \int f(x)\,\mathrm{d}\Psi_t^+(x)/n_t^+$ and $\mathrm{E}_t^-[f(x)] \equiv \int f(x)\,\mathrm{d}\Psi_t^-(x)/n_t^-$ be the averages of a function of position over the units that enter and exit at t. The demography of Proposition 2 is then

$$\text{Demography} = n_t^+ \mathrm{E}_t^+[\mathrm{IF}_\nu(x; G_t)] - n_t^- \mathrm{E}_t^-[\mathrm{IF}_\nu(x; G_t)],$$

the entry rate times the average influence function of entrants, minus the exit rate times that of exiters, so the influence function of each statistic below gives its demography. For inequality indices, entrants' and exiters' positions are normalized by the mean of the population.

For scale- and population-invariant inequality indices, the sign of demography follows from the shape of the influence function. The transfer principle makes IF_ν convex; population invariance makes it average to zero, $\mathrm{E}[\mathrm{IF}_\nu(x_{it}; G_t)] = 0$. The influence function is therefore negative on an interval around the position at which a marginal increase leaves the statistic unchanged, and positive in the tails. Demography raises the statistic when entrants are concentrated in the tails relative to incumbents, or exiters near the middle, and lowers it in the opposite case.

Table E1: Existing Accounting Frameworks for the Change in a Statistic

Statistic	Accounting framework	Drift	Displacement	Demography
Variance	Hart and Prais (1956)	≈	≈	✓
HHI	Kambhampati and Kattuman (2009)	✓	✓	
	Benkard et al. (2026)			✓
Exponential mean	Fernholz and Shay (1982)	≈	≈	
	Feenstra (1994)			✓
Quantile	Berman and Bourguignon (2024)	✓	✓	
Top share	Gomez (2023)	✓	✓	≈
Gini	Jenkins and Van Kerm (2006)	✓	✓	
Weighted average	Foster et al. (2001)	✓	✓	≈

Notes: For each statistic, the table lists the earliest accounting frameworks for its change that I know of. A check means that the framework has a term equal to the component of Section 1.1; ≈ means that the two differ by terms that vanish in continuous time. Drift and displacement are compared in the absence of entry and exit. For the top share, demography is compared for average wealth at the top deflated by mean wealth in the whole population, the statistic that Gomez (2023) decomposes (Appendix E.6.2). A framework with demography but neither drift nor displacement separates entry and exit from the change among continuing firms without splitting it.

E.1 Average and Average of a Function

Statistic. The average of a function f of positions, $\nu(G_t) = \mathrm{E}[f(x_{it})]$, covers the average ($f(x) = x$), moments ($f(x) = x^\alpha$), and expected utility (f a utility function). Over an interval, displacement is the average gap between f at final positions and its tangent at initial positions (Table 1). It is non-negative when f is convex and non-positive when f is concave; with f a concave utility function, it is the loss of average utility from the dispersion of changes among units that start at the same position. The average is linear, so all change among continuing units is drift.

Law of motion. The statistic is population invariant, and its influence function is

$$\mathrm{IF}_\nu(x; G_t) = f(x) - \mathrm{E}[f(x_{it})],$$

with first derivative $f'(x)$ and second $f''(x)$. Adding a unit raises the statistic when its own $f(x)$ exceeds the average of the incumbents' $f(x_{it})$: entry raises the average when entrants lie above the mean of incumbents, and exit when exiters lie below it.

E.2 Variance

Statistic. The variance, $\nu(G_t) = \mathrm{Var}(x_{it})$, is quadratic in positions, so its drift and displacement in Table 1 hold at any horizon. A common shift of all positions leaves it unchanged.

Law of motion. The influence function is

$$\mathrm{IF}_\nu(x; G_t) = (x - \mathrm{E}[x_{it}])^2 - \mathrm{Var}(x_{it}),$$

with first derivative $2(x - \mathrm{E}[x_{it}])$ and second derivative 2. Adding a unit raises the variance when the unit lies more than one standard deviation from the mean, so entry raises the variance when entrants are farther from the mean, in mean square, than incumbents.

Related literature. Hart and Prais (1956) split the change in the variance of log firm size into deaths, the change among continuing firms, and births. As demography does here, they compare deaths with all firms at the start and births with all firms at the end. They write the change among continuing firms through the regression of final on initial log size, as β^2 times the initial variance plus the residual variance. These two terms are the drift and displacement of the variance applied to logs, up to terms of order Δt^2. Campbell et al. (2019) use the same split for the variance of log wealth.

E.3 Herfindahl–Hirschman Index

Statistic. The HHI is defined on the measure of firms, whose mass N_t is the number of firms; it is not population invariant. In terms of firms' sales x, the statistic is

$$\nu(G) = \frac{\int x^2 \, \mathrm{d}G}{\left(\int x \, \mathrm{d}G\right)^2},$$

the sum of the firms' squared shares $s_i = x_i / \sum_j x_j$. Over an interval, written in continuing firms' shares among themselves, s_{it}/s_{Ct}, the statistic is quadratic, so drift is $2\sum_{i\in C}(s_{it}/s_{Ct})\Delta(s_{it}/s_{Ct})$ and displacement $\sum_{i\in C}(\Delta(s_{it}/s_{Ct}))^2$ at any horizon (Section 4.3).

Law of motion. The influence function is

$$\mathrm{IF}_\nu(x; G) = \frac{x^2}{\left(\int z \, \mathrm{d}G(z)\right)^2} - \frac{2\nu(G)x}{\int z \, \mathrm{d}G(z)},$$

which is $s^2 - 2\nu(G)s$ in the share $s = x/\int z\,\mathrm{d}G(z)$ of the added firm. A firm added with share s adds its own squared share and lowers the shares of all other firms in proportion, which lowers the statistic by $2\nu(G)s$: to first order, an entrant raises the statistic only if $s > 2\nu(G)$. The first derivative is $2(s - \nu(G))/\int z\,\mathrm{d}G(z)$, and the second derivative $2/(\int z\,\mathrm{d}G(z))^2$. Under the continuum assumption of Proposition 2, aggregate sales have no idiosyncratic volatility. Writing the integrals as sums,

$$\text{Drift} = 2\sum_i s_{it}^2 \left(\frac{\mu_t(x_{it})}{x_{it}} - \sum_j s_{jt} \frac{\mu_t(x_{jt})}{x_{jt}} \right), \qquad \text{Displacement} = \sum_i s_{it}^2 \frac{\sigma_t(x_{it})^2}{x_{it}^2}.$$

When some firms are large, each firm's shocks also move total sales, which adds to displacement a granular term (Appendix B.3) and, in shares, a term to drift (Appendix D.3). In demography, a representative entrant, drawn from G_t/N_t, contributes $\int \mathrm{IF}_\nu\,\mathrm{d}(G_t/N_t) = -\nu(G_t)/N_t$. Mass (Section 1.1) is therefore $-\nu(G_t)(n_t^+ - n_t^-)/N_t$, negative whenever the number of firms grows, the continuous-time analogue of the $1/N$ effect; selection is zero when entrants and exiters are drawn from the incumbent distribution.

Related literature. Kambhampati and Kattuman (2009) split the change in the HHI among continuing firms into size-related growth and market-share mobility, which are drift and displacement, and Benkard et al. (2026) separate entry, exit, and mergers from the change among continuing firms.

E.4 Generalized Entropy and Atkinson Indices

E.4.1 Generalized Entropy

Statistic. For $\alpha \notin \{0, 1\}$, GE(α) is the α-th moment of normalized positions divided by $\alpha(\alpha - 1)$, net of a constant: $\nu(G_t) = (\mathrm{E}[\tilde{x}_{it}^\alpha] - 1)/(\alpha(\alpha - 1))$. Its limits as $\alpha \to 0$ and $\alpha \to 1$ are the mean log deviation and the Theil index. GE indices are scale invariant, population invariant, and satisfy the transfer principle for all α. Over an interval, their drift and displacement are those of the average of $f(x) = x^\alpha$ applied to $\tilde{x}_{it}$, divided by $\alpha(\alpha - 1)$.

Law of motion. The influence function has one more term than that of the moment, since the added unit also raises the mean position, which lowers every normalized position:

$$\mathrm{IF}_\nu(\tilde{x}; \tilde{G}_t) = \frac{\tilde{x}^\alpha - \mathrm{E}[\tilde{x}_{it}^\alpha] - \alpha \mathrm{E}[\tilde{x}_{it}^\alpha](\tilde{x} - 1)}{\alpha(\alpha - 1)}.$$

The first derivative is $(\tilde{x}^{\alpha-1} - \mathrm{E}[\tilde{x}_{it}^\alpha])/(\alpha - 1)$ and the second $\tilde{x}^{\alpha-2}$, so

$$\text{Drift} = \frac{1}{\alpha - 1}\mathrm{E}[\tilde{x}_{it}^{\alpha-1}\tilde{\mu}_t(\tilde{x}_{it})], \qquad \text{Displacement} = \frac{1}{2}\mathrm{E}[\tilde{x}_{it}^{\alpha-2}\tilde{\sigma}_t(\tilde{x}_{it})^2].$$

The term from the mean drops out of drift, since $\mathrm{E}[\tilde{\mu}_t(\tilde{x}_{it})] = 0$, but not out of demography. Drift weights each unit's change in normalized position by $\tilde{x}_{it}^{\alpha-1}/(\alpha - 1)$: for $\alpha > 1$, gains at

the top raise the statistic most; for $\alpha < 1$, gains at the bottom lower it most. This weight rises with normalized position for every α, so the displacement of every generalized entropy index is non-negative at any horizon.

E.4.2 Theil Index

Statistic. The Theil index (Theil, 1967), $\nu(G_t) = \mathrm{E}[\tilde{x}_{it} \ln \tilde{x}_{it}]$, is the entropy of the shares $s_i = x_i / \sum_j x_j$ relative to equal shares. It is the limit of GE(α) as $\alpha \to 1$. Over an interval, its drift is $\mathrm{E}[\ln \tilde{x}_{it} \cdot \Delta\tilde{x}_{it}]$ and its displacement $\mathrm{E}[\tilde{x}_{i,t+\Delta t} \ln(\tilde{x}_{i,t+\Delta t} / \tilde{x}_{it})]$, the limits of those of GE(α).

Law of motion. The influence function, the limit of the one above as $\alpha \to 1$, is $\mathrm{IF}_\nu(\tilde{x}; \tilde{G}_t) = \tilde{x} \ln \tilde{x} - \mathrm{E}[\tilde{x}_{it} \ln \tilde{x}_{it}] - (1 + \mathrm{E}[\tilde{x}_{it} \ln \tilde{x}_{it}])(\tilde{x} - 1)$, with first derivative $\ln \tilde{x} - \mathrm{E}[\tilde{x}_{it} \ln \tilde{x}_{it}]$ and second $1/\tilde{x}$. Drift is therefore $\mathrm{E}[\tilde{\mu}_t(\tilde{x}_{it}) \ln \tilde{x}_{it}]$ and displacement $\frac{1}{2}\mathrm{E}[\tilde{\sigma}_t(\tilde{x}_{it})^2 / \tilde{x}_{it}]$, the limits of those of GE(α).

E.4.3 Power Mean and Atkinson-Equivalent Wealth

Statistic. The power mean of order $\alpha \neq 0$ is $\nu(G_t) = \mathrm{E}[x_{it}^\alpha]^{\frac{1}{\alpha}}$. With $\alpha = 1 - \epsilon$, it is Atkinson-equivalent wealth: the wealth that, received by everyone, gives the same average utility as the actual distribution when utility has constant relative risk aversion $\epsilon > 0$, Atkinson's inequality aversion (Atkinson, 1970). The statistic is population invariant. Over an interval, drift is $\mathrm{E}[x_{it}^\alpha]^{\frac{1-\alpha}{\alpha}} \mathrm{E}[x_{it}^{\alpha-1} \Delta x_{it}]$ and displacement $\mathrm{E}[x_{i,t+\Delta t}^\alpha]^{\frac{1}{\alpha}} - \mathrm{E}[x_{it}^\alpha]^{\frac{1-\alpha}{\alpha}} \mathrm{E}[x_{it}^{\alpha-1} x_{i,t+\Delta t}]$. The power mean is homogeneous of degree one, so, by Euler's theorem, drift is the statistic times a weighted average of growth rates, $\nu(G_t)\mathrm{E}[x_{it}^\alpha \Delta x_{it} / x_{it}] / \mathrm{E}[x_{it}^\alpha]$, with weights proportional to x_{it}^α: a common growth rate is all drift, and with $\alpha = 1 - \epsilon$ the weights favor poorer units, so drift is pro-poor growth. When $\alpha < 1$, the power mean is concave in positions, so its tangent lies above its final value and displacement is non-positive at any horizon; when $\alpha > 1$, it is convex, and displacement is non-negative.

Law of motion. The influence function is

$$\mathrm{IF}_\nu(x; G_t) = \frac{1}{\alpha} \mathrm{E}[x_{it}^\alpha]^{\frac{1-\alpha}{\alpha}} \left(x^\alpha - \mathrm{E}[x_{it}^\alpha]\right),$$

so an added unit raises the statistic when its position exceeds the power mean, for either sign of α. The first derivative is $\mathrm{E}[x_{it}^\alpha]^{\frac{1-\alpha}{\alpha}} x^{\alpha-1}$ and the second $(\alpha - 1)\mathrm{E}[x_{it}^\alpha]^{\frac{1-\alpha}{\alpha}} x^{\alpha-2}$, so displacement in continuous time also has the sign of $\alpha - 1$. With $\alpha = 1 - \epsilon$ and normalized positions, the power mean is Atkinson-equivalent normalized wealth, one minus the Atkinson index, which therefore has the opposite drift and displacement. Appendix D.2.3 decomposes its log, whose drift is the upward mobility of Genicot and Ray (2023) and whose displacement is the welfare cost of risk.

E.5 Exponential Mean

Statistic. Aggregate productivity and price indices are sums over varieties, so, as for the HHI, the exponential mean is defined on the measure of units, whose mass is their number. The statistic is

$$\nu(G) = \frac{1}{\gamma} \ln \int e^{\gamma x} \, \mathrm{d}G(x).$$

Over an interval, with $s_{it} \equiv e^{\gamma x_{it}} / \sum_j e^{\gamma x_{jt}}$ and $s_{Ct} \equiv \sum_{i \in C} s_{it}$, drift is $\sum_{i \in C} (s_{it}/s_{Ct}) \Delta x_{it}$ and displacement

$$\begin{aligned} \text{Displacement} &= \frac{1}{\gamma} \sum_{i \in C} \frac{s_{it}}{s_{Ct}} \ln \frac{s_{it}/s_{Ct}}{s_{i,t+\Delta t}/s_{C,t+\Delta t}} \\ &= \frac{1}{\gamma} \ln \sum_{i \in C} \frac{s_{it}}{s_{Ct}} e^{\gamma \Delta x_{it}} - \sum_{i \in C} \frac{s_{it}}{s_{Ct}} \Delta x_{it} \\ &= \sum_{n \geq 2} \frac{\gamma^{n-1}}{n!} \kappa_{nt}, \end{aligned}$$

where κ_{nt} is the n-th cumulant of the changes Δx_{it}, each unit weighted by s_{it}/s_{Ct}. The first line is $1/\gamma$ times the relative entropy between continuing units' initial and final weights. The second is $1/\gamma$ times the cumulant generating function of the weighted changes at γ, net of its first term, which is drift. In the third, the variance gives the first term of the sum, so over short intervals displacement is $\gamma/2$ times the weighted variance of the changes; the skewness and higher cumulants of the changes give the others. When $\gamma > 0$, the units whose position rises gain weight, so displacement is positive; when $\gamma < 0$, as for the price index, the goods whose prices rise lose weight, so displacement is negative. Adding λ to every position adds λ to the statistic, so a common shift is all drift.

Law of motion. The influence function is

$$\mathrm{IF}_\nu(x; G) = \frac{e^{\gamma x}}{\gamma \int e^{\gamma z} \, \mathrm{d}G(z)}.$$

A unit added at x raises the statistic by the share of sales it takes, divided by γ. The first derivative is $e^{\gamma x} / \int e^{\gamma z} \, \mathrm{d}G(z)$, the share of sales that a unit at x takes, and the second derivative is γ times the first, so drift is $\sum_i s_{it} \mu_t(x_{it})$ and displacement $\frac{\gamma}{2} \sum_i s_{it} \sigma_t(x_{it})^2$. Since x_{it} is a log value, $\mu_t(x_{it})$ is the expected log growth of the level $e^{x_{it}}$, its expected growth rate minus half the variance of its growth, and $\sigma_t(x_{it})^2$ is the variance of its growth (equation (C1)). Demography is the share of sales that entrants take minus the share that exiters held, per unit of time, divided by γ. The statistic is not population invariant: $\int \mathrm{IF}_\nu \, \mathrm{d}G = 1/\gamma$, so replicating the population changes it by $\ln(\lambda)/\gamma$. When some units are large, their shocks move the sum $\int e^{\gamma x} \, \mathrm{d}G$: a large unit's gain raises the sum, which lowers its own share. Displacement then gains the granular term of Appendix B.3, $-\frac{\gamma}{2} \sum_i s_{it}^2 \sigma_{it}^2$ with independent shocks, where σ_{it} is the volatility of unit i's own shock.

Related literature. With x log productivity, drift is the Hulten term (Hulten, 1978). Fernholz and Shay (1982) write the log growth of the total capitalization of a stock market, the exponential mean with $\gamma = 1$ of the stocks' log capitalizations, as the capitalization-weighted average of their log growth, which is drift, plus the excess growth rate, which is displacement in continuous time. Feenstra (1994) writes the change in the constant-elasticity price index as the change in a price index of the goods sold at both dates, which is the change among continuing units, plus a correction for new and disappearing goods, which is demography.

E.6 Ordered Weighted Averages

Statistic. Ordered weighted averages (OWA) weight positions by an integrable function $w : [0,1] \to \mathbb{R}$ of the percentile rank $R_{it} = R_t(x_{it})$, where the rank function $R_t(x)$ is the share of units with position at most x: $\nu(G_t) = \mathrm{E}[w(R_{it})x_{it}]$. They are population invariant and homogeneous of degree one; they satisfy the transfer principle if and only if $w(\cdot)$ is weakly increasing. Over an interval, displacement (Table 1) is the re-ranking of Section 4.2.

Law of motion. The influence function is

$$\mathrm{IF}_\nu(x; G_t) = W_t(x) - \mathrm{E}[W_t(x_{it})], \qquad W_t(x) \equiv \int_0^x w(R_t(y))\,\mathrm{d}y.$$

Each increment of the added unit's position counts at the rank weight of the level it reaches; the constant makes the influence function average to zero. The first derivative is $w(R_t(x))$ and the second derivative $w'(R_t(x))g_t(x)$, where g_t is the density of positions.

Normalized positions. Applied to normalized positions $\tilde{x}_{it}$, the same weights give the relative Gini and the top p share divided by p. The mean has no displacement, and writing drift with $\tilde{\mu}_t$ nets out its drift, so drift and displacement keep the formulas of Table 2, in $\tilde{x}_{it}$, $\tilde{\mu}_t$, $\tilde{\sigma}_t$, and the density $\tilde{g}_t$ of normalized positions. The demography of the mean remains: the influence function is that of the OWA minus $\nu(G_t)$ times that of the mean, $\tilde{x} - 1$ in normalized positions.

E.6.1 Quantiles

Statistic. The p-quantile is $\nu(G_t) = q_{pt} \equiv R_t^{-1}(p)$, an OWA whose weight is a point mass at p. Over an interval, this point mass leaves drift undefined in panel data (Appendix A), while the change among continuing units and demography are defined.

Law of motion. With g_t the density of positions, the influence function is

$$\mathrm{IF}_\nu(x; G_t) = -\frac{\mathbf{1}_{x \le q_{pt}} - p}{g_t(q_{pt})}.$$

Its derivatives, in the sense of distributions (Assumption A2(b)), are $\delta_{q_{pt}}/g_t(q_{pt})$ and $\delta'_{q_{pt}}/g_t(q_{pt})$, so

$$\text{Drift} = \mu_t(q_{pt}), \qquad \text{Displacement} = -\frac{1}{2}\frac{\partial_x(g_t(q_{pt})\sigma_t(q_{pt})^2)}{g_t(q_{pt})}.$$

Drift is the expected change in position of the units at the quantile. Displacement is the net flow of units across the quantile caused by volatility: shocks move units from where $g_t\sigma_t^2$ is high to where it is low, so, with constant volatility, they raise quantiles above the mode and lower those below. Entry raises the quantile when fewer than a share p of entrants are below it, and exit when more than a share p of exiters are. For the log quantile, each component is divided by q_{pt}; its drift, $\mu_t(q_{pt})/q_{pt}$, is the non-anonymous growth incidence curve.

Smoothed quantiles. The smoothed p-quantile replaces the point mass by an approximate one, $w(R) = \delta^h_{R-p}$, with $\delta^h_u \equiv K(u/h)/h$ for a kernel K that is a density on $[-1,1]$ and a bandwidth $0 < h \leq \min(p, 1-p)$: $\nu(G_t) = \mathrm{E}[\delta^h_{R_{it}-p}x_{it}]$. Over an interval, drift is $\mathrm{E}[\delta^h_{R_{it}-p}\Delta x_{it}]$ and displacement $\mathrm{E}[x_{i,t+\Delta t}\Delta\delta^h_{R_{it}-p}]$, those of the OWA (Table 1). With the uniform kernel, $\delta^h_{R-p} = \mathbf{1}_{|R-p|\leq h}/(2h)$, it is the average position of the units between the $p-h$ and $p+h$ percentiles. Unlike that of the quantile, its influence function has no jumps, so drift is defined in panel data (Assumption A3). With the uniform kernel, drift over an interval is the average change in position of the units initially in the band, and displacement is the total final position of the units that enter the band minus that of the units that leave it, divided by the number of units in the band. In continuous time, with the uniform kernel, displacement is $((g_t\sigma_t^2)(q_{p-h,t}) - (g_t\sigma_t^2)(q_{p+h,t}))/(4h)$: at the lower edge, as at the threshold of a top group, the units that enter the band have moved up and those that leave it have moved down, which raises the statistic; the upper edge does the reverse. As $h \to 0$, the three terms converge to those of the quantile.

Related literature. Berman and Bourguignon (2024) compare two growth incidence curves on a balanced panel. The anonymous curve is the growth of each quantile, its total change. The non-anonymous curve is the average growth of the units initially at each quantile, its drift. The gap between the two is displacement.

E.6.2 Top Average

Statistic. For the top p average, $w(R) = \mathbf{1}_{R\geq 1-p}/p$, so the statistic is the average position of the top group, $\nu(G_t) = \mathrm{E}[x_{it} \mid R_{it} \geq 1-p]$. Over an interval, drift is the average change of the initial top group, and displacement is the final average of the final top group minus that of the initial top group: the units that enter the top group end above those that leave it.

Law of motion. With $q_{1-p,t}$ the threshold of the top group, $W_t(x) = (x - q_{1-p,t})^+/p$, so the influence function is

$$\mathrm{IF}_\nu(x; G_t) = \frac{(x-q_{1-p,t})^+}{p} + q_{1-p,t} - \nu(G_t).$$

An added unit above the threshold adds its excess over it, spread over the top group; wherever it is added, it enlarges the top group by a fraction p of its mass, taken in at the threshold, which lowers the average. An entrant below the threshold thus lowers the statistic by $\nu(G_t) - q_{1-p,t}$, the gap between the top average and the threshold. The second derivative is a point mass at $q_{1-p,t}$ divided by p, so displacement is half the variance of the shocks of units at the threshold, times their density, divided by p: shocks move as many units into the top group as out of it, but those that enter have moved up and those that leave have moved down.

Top share. The top p share is p times the top p average of normalized positions. Over an interval, drift is the change in the share of the total held by the initial top group, and displacement the final share held by the units that enter the top group minus that held by the units that leave it. With $\tilde{q}_{1-p,t}$ the normalized position at the threshold, the influence function is p times that of the top average, in normalized positions, minus $\nu(G_t)$ times that of the mean, $\tilde{x} - 1$,

$$\mathrm{IF}_\nu(\tilde{x}; \tilde{G}_t) = (\tilde{x} - \tilde{q}_{1-p,t})^+ + p\tilde{q}_{1-p,t} - \nu(G_t)\tilde{x},$$

since an added unit also raises the mean, which lowers the share in proportion to its normalized position. Drift, $\mathrm{E}[\mathbf{1}_{R_{it} \geq 1-p} \tilde{\mu}_t(\tilde{x}_{it})]$, is the top share times the gap between the expected growth rate of the top group's wealth and that of total wealth. Displacement is p times that of the top p average, divided by mean wealth: $\frac{1}{2}\tilde{g}_t(\tilde{q}_{1-p,t})\tilde{\sigma}_t(\tilde{q}_{1-p,t})^2$. For the log top share, the components over an interval follow from the log of the split (Appendix C.2); in continuous time, they are those of the top share divided by the top share.

Related literature. The decomposition of Gomez (2023) can be interpreted as the decomposition here applied to the growth of average wealth at the top (Appendix C.2), with wealth deflated by mean wealth in the whole population, $\bar{x}_t$. The deflator is not part of the statistic but is treated as exogenous, so the effect of entrants and exiters on mean wealth counts in drift rather than in demography. This choice maps the components directly into the growth rate, volatility, and death rate of random growth models, and requires observing only the top group. For the same initial top group, the drift of that paper is

$$\mathrm{Drift}^{\ln \nu} + \ln \frac{\bar{x}^C_{t+\Delta t}}{\bar{x}^C_t} - \ln \frac{\bar{x}_{t+\Delta t}}{\bar{x}_t},$$

the drift here plus the log growth of mean wealth among continuing households, $\bar{x}^C_t$, minus that in the whole population. The gap vanishes when exiters and entrants are, on average, as rich as continuing households. Otherwise it remains in continuous time, where it equals $-n^+_t(\mathrm{E}^+_t[\tilde{x}] - 1) + n^-_t(\mathrm{E}^-_t[\tilde{x}] - 1)$, minus the effect of entry and exit on log mean wealth. That paper also measures demography at the end of the interval rather than at its start, which matters only at second order in Δt.

E.6.3 Gini Coefficient

Statistic. For the Gini, $w(R) = 2R - 1$, so the absolute Gini is $\nu(G_t) = \mathrm{E}[(2R_{it} - 1)x_{it}]$. Over an interval, drift is $2\mathrm{Cov}(R_{it}, \Delta x_{it})$, twice the covariance between initial rank and the change in position, and displacement, the re-ranking of Section 4.2, is $2\mathrm{E}[x_{i,t+\Delta t}\Delta R_{it}]$, approximately $\mathrm{E}[\Delta R_{it}\Delta x_{it}]$. Since $2R - 1 = 2\int_0^1 \mathbf{1}_{R \geq 1-p}\,\mathrm{d}p - 1$, the absolute Gini is twice the integral over p of p times the top p average, minus the mean position. Each component of its change is therefore twice the integral over p of p times the corresponding component of the top p average, minus that of the mean, which has no displacement.

Law of motion. The absolute Gini is also half the average absolute gap in positions between two units drawn independently, which gives its influence function a simple form:

$$\mathrm{IF}_\nu(x; G_t) = \mathrm{E}[|x - x_{it}|] - 2\nu(G_t).$$

An added unit raises the Gini when its average gap to incumbents exceeds the average gap between two incumbents, $2\nu(G_t)$. The second derivative is $2g_t(x)$, so displacement weights volatility by the density: the closer units are, the more they are re-ordered when shocks hit.

Relative Gini. The usual, relative, Gini is the absolute Gini divided by mean wealth. Since normalized positions average one, it is twice the integral of the top p shares over p minus one, so each component of its change is twice the integral over p of the corresponding top-share component. Its influence function is $\mathrm{E}[|\tilde{x} - \tilde{x}_{it}|] - \nu(G_t)(1 + \tilde{x})$, so an entrant richer than the mean also lowers it by raising the mean.

Related literature. Jenkins and Van Kerm (2006) write the change in the Gini on a panel of continuing individuals as re-ranking minus progressivity, which are displacement and minus drift.

E.7 Weighted Averages

Statistic. A weighted average is a statistic of two variables. Let x denote a firm's productivity and s its share of the size of all firms. Average productivity weighted by size is $\nu(G) = \int sx\,\mathrm{d}G / \int s\,\mathrm{d}G$; the denominator is one, but it makes the statistic depend on shares only through their ratios, so that it is defined for any set of firms. Among continuing firms, $\nu(G_t^C) = \sum_{i \in C}(s_{it}/s_{Ct})x_{it}$. The weight on a firm's productivity is its share; the weight on its share is its productivity, measured relative to the statistic since shares sum to one at both dates. Proposition 1, applied to productivity and shares, gives

$$\Delta\nu(G_t^C) = \underbrace{\sum_{i \in C}\frac{s_{it}}{s_{Ct}}\Delta x_{it} + \sum_{i \in C}\Delta\frac{s_{it}}{s_{Ct}}\left(x_{it} - \nu(G_t^C)\right)}_{\text{Drift}} + \underbrace{\sum_{i \in C}\Delta\frac{s_{it}}{s_{Ct}}\Delta x_{it}}_{\text{Displacement}}.$$

The statistic is linear in each variable, so displacement comes only from how a firm's productivity and share move together: it is positive when the firms whose productivity rises gain share and negative when they lose it. Entry is the entry share times the gap between the average productivity of entrants and that of continuing firms; exit is the exit share times the gap between the average productivity of continuing firms and that of exiters.

Law of motion. The influence function, $s(x - \nu(G))/\int s\,\mathrm{d}G$, is linear in each variable, so in continuous time drift and displacement are those of the average product in Table 2, with $y = s$ (equation (B1)): displacement depends on the covariance between productivity and size shocks rather than on the variance of either, so it can have either sign.

Related literature. Foster et al. (2001) split the change in the share-weighted average of firms' productivity into within, between, and cross terms: the within and between terms are its drift; the cross term is its displacement. Their entry and exit terms compare entrants and exiters with the statistic of all firms at the initial date, which biases them (Melitz and Polanec, 2015); the entry and exit terms here compare them with continuing units, as Melitz and Polanec (2015) do. These decompositions have since been applied to other weighted averages. De Loecker et al. (2020) split the change in the sales-weighted average markup into within, reallocation, cross, and net entry terms, as Foster et al. (2001) do for productivity: their cross term is displacement. Autor et al. (2020) apply the decomposition of Melitz and Polanec (2015) to the labor share. Foster et al. (2001) note that, with output weights, noise in output makes the cross term spuriously positive and the within term spuriously low: this is the mean reversion of Section 3. The decomposition of Griliches and Regev (1995), which they also report, evaluates the weights and the productivity gaps at the average of their initial and final values, which moves the bias into drift rather than removing it; the lagged weights of Section 3 remove it.

Industry productivity. With productivity in logs, $x_{it} = \ln A_{it}$, the weighted average differs from industry productivity, the statistic of Section 4.1, by the entropy of sales shares. Consider an industry k whose firms are the same at both dates, with sales proportional to $A_{it}^{\theta-1}$ as in (4), so that $\ln A_{it} = \ln \bar{A}_{kt} + \ln s_{it}/(\theta - 1)$. Averaging with sales shares gives

$$\sum_{i \in C_k} s_{it} \ln A_{it} = \ln \bar{A}_{kt} - \frac{H_{kt}}{\theta - 1}, \qquad H_{kt} \equiv -\sum_{i \in C_k} s_{it} \ln s_{it}.$$

Industry productivity rewards sales spread over many firms, the love of variety of Section 4.1; the weighted average does not. The within term of Foster et al. (2001) is therefore the drift of industry productivity, the Hulten term, while their between and cross terms add up to its displacement minus $\Delta H_{kt}/(\theta - 1)$. Substituting for $\ln A_{it}$, with $\sum_{i \in C_k} \Delta s_{it} = 0$, writes each of

the two terms with shares alone:

$$\text{Between} \equiv \sum_{i \in C_k} \Delta s_{it} \ln A_{it} = \frac{1}{\theta - 1} \sum_{i \in C_k} \Delta s_{it} \ln s_{it},$$

$$\text{Cross} \equiv \sum_{i \in C_k} \Delta s_{it} \Delta \ln A_{it} = \text{Displacement} + \frac{1}{\theta - 1} \sum_{i \in C_k} s_{i,t+\Delta t} \ln \frac{s_{i,t+\Delta t}}{s_{it}},$$

where displacement is that of industry productivity, the relative entropy of (4). The last sum is the relative entropy with the two dates exchanged, equal to displacement to second order. With productivity inferred from shares, the cross term is therefore about twice displacement. Dispersion in firms' changes concentrates sales, which the weighted average counts as a further gain and industry productivity offsets by the loss of variety.

E.8 Other Statistics

Many statistics are smooth functions of those above. The influence function satisfies the chain rule, and the three terms of Proposition 2 are linear in it, so a statistic $f(\nu_1, \ldots, \nu_K)$ has $\text{Drift}(f) = \sum_k (\partial f / \partial \nu_k) \text{Drift}(\nu_k)$, and likewise for displacement and demography. Over an interval, drift satisfies this rule exactly, and displacement and demography up to terms of order Δt^2 (Appendix C.2). This gives, for example, the components of the interquartile range and quantile ratios, the inverted Pareto coefficient (Blanchet et al., 2022), the Hill estimator of the tail exponent (Hill, 1975), skewness and kurtosis, and the slope of a cross-sectional regression. The poverty measures of Foster et al. (1984) are averages of a function that is convex for $\alpha \geq 1$, so their displacement is non-negative; the headcount ratio, like a quantile, requires smoothing over an interval.

Appendix References

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