Q-Theory

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Investment Problem

Hayashi (1982) Set Up

- The firm produces according to technology.

$$F(K_t)$$

- The evolution of capital is

$$dK_t = (I_t - \delta K_t)dt$$

- The firm profit $Y_t dt$ is

$$Y_t dt = F(K_t)dt - I_t dt - C_t(I_t, K_t)dt$$

Assume $C_I > 0$ and $C_{II} > 0$ (convex costs)

Hayashi (1982) Marginal q

- Denote V_t the value of the firm. The firm takes the interest rate as given and chooses investment I_t to maximize its value:

$$V_t = \max_{(I_s)} E_t\left[\int_{s\geq t} e^{-r(s-t)} Y_s ds\right]$$
$$dK_t = (I_t - \delta K_t)dt$$

- The HJB corresponding to this problem is

$$0 = \max_{I_t} \{Y_t dt + E_t[dV_t] - rV_t dt\}$$

Applying Ito's Lemma:

$$0 = \max_{I_t} \{(F(K_t) - I_t - C_t(I_t, K_t))dt - rV_t dt + (I_t - \delta K_t)V_K dt\}$$

- FOC with respect to I_t gives

$$1 + C_I(I_t, K_t) = V_K$$

V_K is called "marginal q"

Hayashi (1982) Marginal q

- To get more intuition on the economic meaning of the marginal ‘q‘, derive HJB w.r.t. K to obtain

$$0 = (F_K(K_t) - C_K(I_t, K_t))dt - rV_K dt - \delta V_K dt + (I_t - \delta K_t)V_{KK} dt$$

- This can be written as

$$0 = (F_K(K_t) - C_K(I_t, K_t))dt - (r + \delta)V_K dt + E[dV_K]$$

  In integrated form (note the similarity with HJB)

$$V_K = E\left[\int_t^\infty e^{-(r+\delta)(s-t)}(F_K(K_s) - C_K(I_s, K_s))ds\right]$$

- The marginal q is how much the firm values having one supplementary unit of capital. It is the discounted payoff of having one ore unit of installed capital plus any savings in investment costs due to a marginally bigger plant size.

Average Q

- Now, assume that
  1. Production function is homogeneous of degree one in K
  $$F(K_t, L_t) = A_t K_t$$
  2. firm's adjustment cost is homogeneous of degree one in I and K
  $$C(I_t, K_t) = c_t(i_t) K_t$$
  $$i_t \equiv \frac{I_t}{K_t}$$
  In particular, profits and law of motion of capital become linear in capital
  $$Y_t dt = (A_t - i_t - c(i_t)) K_t dt$$
  $$K_t = (i_t - \delta) K_t dt$$
- The value function is now linear in K: $V_t(K_t) = Q_t K_t$. The first order condition gives
  $$1 + c_t'(i_t) = Q_t$$
  The "marginal q" equals the "average q". It is observable as Market-to-Book ratio.

Testing Q-theory

Investment Cashflow sensitivity

- q theory says

$$1 + c_t'(i_t) = Q_t$$
$$\Rightarrow i_t = c_t'^{-1}(Q_t - 1)$$

- Test of Financial friction:

$$i_t = \alpha + \beta_Q \times Q_t + \beta_C \times \text{Cash Flow}_t + \epsilon_t$$

  where Q is the market-to-book ratio (i.e. the average Q).
  - Frictionless model: $\beta_C = 0$
  - Financial frictions: $\beta_C > 0$
- Mismeasurement error: BM_t is not exaclty the marginal q. Cash flow may give some info about marginal q beyond BM_t

Accounting for Financial Frictions

Three ingredients

1. Firm profits are stochastic
2. External financing has fixed cost (information asymetry). When firm has a payout $dU < 0$, cost $g(dU)$.
3. Cash hold by the firms earns less than the risk free rate (agency cost due to free cash, tax distortions, etc)

$\Rightarrow$ firm manages a cash buffer to avoid financial friction

- Firm profits is

$$dY_t = AK_t dt + \sigma K_t dZ_t - I_t dt - C(I_t, K_t) dt$$

- Denote λ the carry cost of cash, and dU_t the firm's payout during dt

$$dW_t = (r - \lambda) W_t dt + dY_t - dU_t$$

Bolton, Chen, Wang (2011)

- The firm problem is to choose investment I and payout U to maximize its value:

$$V(K_t, W_t) = \max_{I,U} E[\int_t^{+\infty} e^{-r(s-t)}(dU_s - g(dU_s))$$
$$dW_t = (r - \lambda)W_t dt + dY_t - dU_t$$
$$dK_t = (I_t - \delta K_t)dt$$

  where $g(dU_t)$ denotes the cost of payout dU_t.

- Assume $g(dU_t) = \phi K + \gamma K dU_t$ if $dU_t \leq 0$, 0 otherwise

- Guess that firm has negative payout (issue shares) when $W = 0$, has positive payout if W is high enough, otherwise does not give money to shareholders.
- HJB between these two boundaries is:

$$
\begin{aligned}
0 &= \max_I \{E[dV] - rV\} \\
&= \max_I \left\{ -rV + (I - \delta K)V_K + ((r - \lambda)W + AK - I - C(I, K))V_W + \frac{\sigma^2}{2} K V_{WW} \right\}
\end{aligned}
$$

- FOC for investment gives

$$
C_I(I_t, K_t) = \frac{V_K}{V_W} - 1
$$

- Denote $w = W/K$ and $i = I/K$. Assume that $C_t(I_t, K_t) = c_t(i_t)K_t$. Guess

$$V(K_t, W_t) = K_t p(w_t)$$

- The problem can be rewritten as:

$$0 = \max_{i*} \left\{ -rp + (i^* - \delta)(p - p'(w)w) + ((r - \lambda)w + A - i^* - c(i^*))p'(w) + \frac{\sigma^2}{2} p''(w) \right\}$$

$$1 + c'(i^*) = \frac{p(w)}{p'(w)} - w$$

Lower Boundary

- The firm raises external fund when cash is $w = 0$. It chooses to raise a quantity m
  1. At this point value of the firm must be continuous before and after raising fund(“value matching” condition)

$$\begin{aligned} p(0) &= -m - g(m) + p(m) \\ &= -\phi - (1+\gamma)m + p(m) \end{aligned}$$

  2. m is chosen until value of one supplementary dollar is cost of capital (“smooth pasting” condition)

$$0 = -1 - \gamma + p'(m)$$

Upper Boundary

- The firm pays shareholders when $w = \overline{w}$
  1. At this point value must be continuous ("value matching" condition)

     $$p(\overline{w}) = p(\overline{w} + dw) - dw$$

     This gives

     $$p'(\overline{w}) = 1$$

  2. Moreover, dw is chosen until value of one supplementary dollar is cost of capital ("smooth pasting" condition)

     $$p'(\overline{w} + dw) = 1$$

     This gives

     $$p''(\overline{w}) = 0$$

- In case of zero or infinitesimal cost, we have condition on first and second derivative, rather than condition on zero and first derivative (like in lower boundary).

Solution

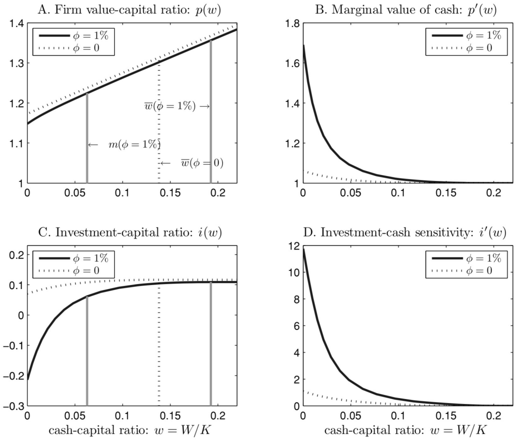

Figure 3. Case II—optimal refinancing. This figure plots the solution for the case of refinancing.