Portfolio Problem

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Complete Markets

- Suppose there are N source of aggregate risk, i.e. $\mathbf{Z}_t = (Z_{1t}, \ldots Z_{Nt})$ is a vector of independent Brownian Motions
- Beyond the risk-free rate, there are N assets with returns given by:

$$\frac{dR_{1t}}{R_{1t}} = \mu_{R1t}dt + \sigma'_{R1t}d\mathbf{Z}_t$$
$$\ldots$$
$$\frac{dR_{Nt}}{R_{Nt}} = \mu_{RNt}dt + \sigma'_{RNt}d\mathbf{Z}_t$$

- We can write the set of returns in a vector form:

$$\frac{d\mathbf{R}_t}{\mathbf{R}_t} = \mu_{Rt}dt + \mathbf{\Sigma}'_t d\mathbf{Z}_t$$

where $\mathbf{R}_t = (R_{1t}, \ldots, R_{Nt})$, $\mu_{Rt} = (\mu_{R1t}, \ldots, \mu_{Rnt})$, and $\mathbf{\Sigma}$ is the matrix such that its j-th column is σ_{Rj}.

- Suppose an investor invests a share of wealth α_{1t} in the first asset, α_{2t} in the second assets, ..., α_{Nt} in the last asset. His/her return is

$$\frac{dR_t}{R_t} = (r_t + \alpha_t'(\mu_{Rt} - r_t))dt + (\Sigma_t \alpha_t)' d\mathbf{Z}_t$$

  where $\alpha = (\alpha_1, \ldots, \alpha_N)$
- Assume that the market is complete, i.e. that Σ_t is invertible. This means that the investor can create a portfolio with an arbitrary exposure σ_t on aggregate shocks $\mathbf{Z}_t$ (using portfolio shares $\alpha_t = \Sigma_t^{-1}\sigma_t$).
- In this case, we can define $\kappa_t = \Sigma_t^{-1}(\mu_{Rt} - r_t)$ the market price of risk. Using this definition, we can write the return of a portfolio with an exposure σ_R on aggregate shocks as:

$$\frac{dR_t}{R_t} = (r_t + \sigma_{Rt}'\kappa_t)dt + \sigma_{Rt}' d\mathbf{Z}_t$$

- Let us examine an asset with a dividend process

$$\frac{dD_t}{D_t} = \mu_{Dt} dt + \sigma_{Dt} dZ_t$$

- Denote $V_t = P_t / D_t$ the price-dividend ratio of this asset. We have

$$\begin{aligned}\frac{dR_t}{R_t} &= \frac{1}{V_t} dt + \frac{d(D_t V_t)}{D_t V_t} \\ &= \underbrace{\left(\frac{1}{V_t} + \mu_{Dt} + \mu_{Vt} + \sigma_{Dt}\sigma_{Vt} \right)}_{\mu_{Rt}} dt + \underbrace{(\sigma_{Dt} + \sigma_{Vt})}_{\sigma_{Rt}} dZ_t\end{aligned}$$

- Market pricing gives

$$\frac{1}{V_t} + \mu_{Dt} + \mu_{Vt} + \sigma_{Dt}\sigma_{Vt} = r_t + \kappa_t'(\sigma_{Dt} + \sigma_{Vt})$$

Static Portfolio Problem

HJB

We first solve the portfolio problem assuming that r_t and κ_t are constant over time

- The investor problem is

$$J_t = \max_{C, \sigma_W} \boldsymbol{E}_t \left\{ \int_t^{+\infty} e^{-\rho(s-t)} U(C_s) ds \right\}$$

$$\frac{dW_t}{W_t} = (r + \sigma'_{Wt}\kappa - \frac{C_t}{W_t})dt + \sigma'_{Wt} dZ_t$$

  Because the market is complete, choosing portfolio shares α is equivalent to choosing a given exposure to aggregate shocks σ_W

- The corresponding HJB is

$$0 = \max_{C_t, \alpha_t} \{U(C_t)dt + E_t[dJ_t] - \rho J_t dt\}$$

CRRA utilities.

- Assume CRRA utilities

$$U(C) = \frac{C^{1-\gamma}}{1-\gamma}$$

- In this case, guess that the value function is

$$J_t = \frac{(W_t \xi)^{1-\gamma}}{1-\gamma}$$

- The HJB equation can be written as

$$0 = \max_{C_t, \alpha_t} \left\{ \frac{C_t^{1-\gamma}}{1-\gamma} + E[\frac{d(W_t \xi)^{1-\gamma}}{1-\gamma}] - \rho \frac{(W_t \xi)^{1-\gamma}}{1-\gamma} \right\}$$

t

FOC

- Substituting the law of motion of wealth:

$$0 = \max_{C_t, \alpha_t} \left\{ \frac{C_t^{1-\gamma}}{W_t^{1-\gamma} \xi^{1-\gamma}} + (1-\gamma) \left(r + \sigma_W' \kappa - \frac{C_t}{W_t} - \frac{\gamma}{2} \sigma_W^2 \right) - \rho \right\}$$

- The FOCs are:

$$[C] : C_t = \xi^{1-\frac{1}{\gamma}} W_t \tag{1}$$

$$[\sigma_W] : \sigma_W = \frac{\kappa}{\gamma} \tag{2}$$

- Plugging the FOCs into HJB we obtain:

$$\xi^{1-\frac{1}{\gamma}} = \frac{\rho}{\gamma} + (1 - \frac{1}{\gamma})\left(r + \frac{1}{2\gamma}\kappa^2\right) \tag{3}$$

  Note that ξ increases with r or κ.

- Remember that

$$\frac{C_t}{W_t} = \xi^{1-\frac{1}{\gamma}}$$

  1. Wealth-to-consumption ratio depends on risk-free rate and market price of risk (does not depend on the current level of consumption).
  2. When $\gamma = 1$, wealth-to-consumption ratio does not depend on r or κ. This is because the income effect of a higher interest rate is compensated by a substitution effect.

CAPM

- Take a return process with drift μ_R and volatility σ_R

$$\frac{dR_t}{R_t} = \mu_R dt + \sigma_R dZ_t$$

Using definition of market price of risk and FOC for σ_{Wt}, we obtain the CAPM:

$$\begin{aligned} \mu_R - r &= \sigma_R' \kappa_t \\ &= \gamma \sigma_R' \sigma_W \end{aligned}$$

Portfolio Problem with Time Varying Investment Opportunities

Asset Returns

- Now let us examine a portfolio problem with time-varying investment opportunities. More precisely, suppose there is an aggregate state variable x_t

$$dx_t = \mu(x_t)dt + \sigma(x_t)dZ_t$$

and that the risk free rate and the market price of risk depend on x_t

$$r_t = r(x_t)$$
$$\kappa_t = \kappa(x_t)$$

- In this case, guess that the value function is

$$J_t = \frac{(W_t \xi(x_t))^{1-\gamma}}{1-\gamma}$$

Using Ito's lemma, ξ_t follows a diffusion with geometric drift μ and geometric volatility $\sigma_{\xi t}$

$$\frac{d\xi_t}{\xi_t} = \underbrace{\frac{\xi'(x_t) + \frac{1}{2}\xi''(x_t)}{\xi(x_t)}}_{\mu_{\xi t}} dt + \underbrace{\frac{\xi'(x_t)}{\xi(x_t)}}_{\sigma_{\xi t}} dZ_t$$

FOC

- The HJB equation can be written as

$$0 = \max_{C_t, \alpha_t} \left\{ \frac{C_t^{1-\gamma}}{1-\gamma} + E[\frac{d(W_t \xi_t)^{1-\gamma}}{1-\gamma}] - \rho \frac{(W_t \xi_t)^{1-\gamma}}{1-\gamma} \right\}$$

- Dividing by the value function and applying Ito's Lemma

$$0 = \max_{C_t, \sigma_{Wt}} \left\{ \frac{C_t^{1-\gamma}}{W_t^{1-\gamma} \xi_t^{1-\gamma}} + (1-\gamma) \left(r_t + \sigma'_{Wt} \kappa_t - \frac{C_t}{W_t} + \mu_{\xi t} - \frac{\gamma}{2} \left(\sigma_{Wt}^2 + \sigma_{\xi t}^2 + 2 \left(1 - \frac{1}{\gamma} \right) \sigma_{Wt} \sigma_{\xi t} \right) \right) - \rho \right\}$$

- The FOCs are:

$$[C] : C_t = \xi_t^{1-\frac{1}{\gamma}} W_t \tag{4}$$

$$[\sigma_{Wt}] : \sigma_{Wt} = \underbrace{\frac{\kappa_t}{\gamma}}_{\text{myopic demand}} + \underbrace{\frac{1-\gamma}{\gamma} \sigma_{\xi t}}_{\text{int. hedging demand}} \tag{5}$$

In the case of $\gamma = 1$, investors do not care about hedging investment opportunities

- Plugging the FOCs into HJB we obtain:

$$\xi_t^{1-\frac{1}{\gamma}} = \frac{\rho}{\gamma} + \left(1 - \frac{1}{\gamma} \right) \left(r_t + \frac{1}{2\gamma} \kappa_t^2 + \frac{(\frac{1}{\gamma} - 2)}{2} \sigma_{\xi t}^2 + (\frac{1}{\gamma} - 1) \sigma_{\xi t} \kappa_t + \mu_{\xi t} \right) \tag{6}$$

This is an ODE on $\xi(x_t)$. Given the functions $\kappa()$ and $r()$, it can be solved to get ξ and compute optimal policy functions

- Take a return process with drift μ_{Rt} and volatility σ_{Rt}

$$\frac{dR_t}{R_t} = \mu_{Rt} dt + \sigma_{Rt} dZ_t$$

Using definition of market price of risk and FOC for σ_{Wt}, we obtain the Intertemporal CAPM:

$$\begin{aligned}\mu_{Rt} - r_t &= \sigma'_{Rt}\kappa_t \\ &= \gamma\sigma'_{Rt}\sigma_{Wt} + (\gamma - 1)\sigma'_{Rt}\sigma_{\xi t}\end{aligned}$$

Euler Equations

Euler Equation

- Applying Ito's lemma on $C_t = \xi_t^{1-\frac{1}{\gamma}} W_t$, we get

$$\sigma_{Ct} = \sigma_{Wt} + \frac{\gamma - 1}{\gamma}\sigma_{\xi t}$$

$$\mu_{Ct} = \mu_{Wt} + \frac{\gamma - 1}{\gamma}\sigma_{\xi t} - \frac{\gamma - 1}{2\gamma^2}\sigma_{\xi t}^2 + \frac{\gamma - 1}{\gamma}\sigma_{Wt}\sigma_{\xi t}$$

  where μ_{Ct}, σ_{Ct} denote the geometric drift and volatility of consumption
- Using FOC for σ_{Wt} and the law of motion of wealth with HJB, we obtain Euler Equations:

$$\sigma_{Ct} = \frac{\kappa_t}{\gamma} \tag{7}$$

$$\mu_{Ct} = \frac{r_t - \rho}{\gamma} + \frac{1 + \gamma}{2\gamma^2}\kappa_t^2 \tag{8}$$

  Note that μ_{Ct}, σ_{Ct} do not depend on σ_ξ
- Denoting $V_t = W_t / C_t$ the wealth-to-consumption ratio:

$$\frac{1}{V_t} + \mu_{Ct} + \mu_{Vt} + \sigma_{Ct}\sigma_{Vt} = r_t + \kappa_t(\sigma_{Ct} + \sigma_{Vt}) \tag{9}$$

- Equations (7), (8) and (9) is an alternative way to solve the portfolio problem

Consumption CAPM

- Take a return process with drift μ_{Rt} and volatility σ_{Rt}

$$\frac{dR_t}{R_t} = \mu_{Rt} dt + \sigma_{Rt} dZ_t$$

Using definition of market price of risk and FOC for σ_{Wt}, we obtain the Consumption CAPM:

$$\mu_{Rt} - r_t = \gamma \sigma_{Ct}' \sigma_{Rt}$$

Testing Euler Equations

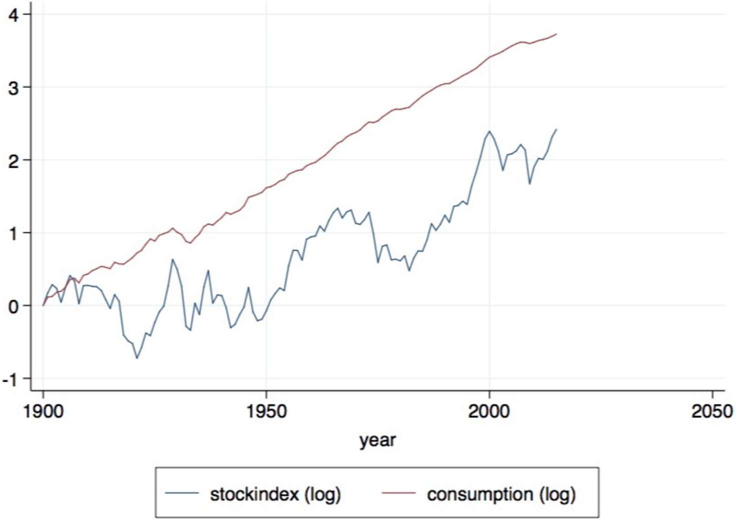
Puzzles

Table 1
International stock and bill returns

Country	Sample period	$\overline{r_e}$	$\sigma(r_e)$	$\rho(r_e)$	$\overline{r_f}$	$\sigma(r_f)$	$\rho(r_f)$
USA	1947.2–1998.4	8.085	15.645	0.083	0.896	1.748	0.508
AUL	1970.1–1999.1	3.540	22.699	0.005	2.054	2.528	0.645
CAN	1970.1–1999.2	5.431	17.279	0.072	2.713	1.855	0.667
FR	1973.2–1998.4	9.023	23.425	0.048	2.715	1.837	0.710
GER	1978.4–1997.4	9.838	20.097	0.090	3.219	1.152	0.348
ITA	1971.2–1998.2	3.168	27.039	0.079	2.371	2.847	0.691
JAP	1970.2–1999.1	4.715	21.909	0.021	1.388	2.298	0.480
NTH	1977.2–1998.4	14.070	17.228	−0.030	3.377	1.591	−0.085
SWD	1970.1–1999.3	10.648	23.839	0.022	1.995	2.835	0.260
SWT	1982.2–1999.1	13.744	21.828	−0.128	1.393	1.498	0.243
UK	1970.1–1999.2	8.155	21.190	0.084	1.301	2.957	0.478
USA	1970.1–1998.4	6.929	17.556	0.051	1.494	1.687	0.571
SWD	1920–1998	7.084	18.641	0.096	2.209	5.800	0.710
UK	1919–1998	7.713	22.170	−0.023	1.255	5.319	0.589
USA	1891–1998	7.169	18.599	0.047	2.020	8.811	0.338

- Average real returns on stocks are high and have high annualized volatility.
- Average real returns on short-term bills low and have low annualized volatility.

Puzzles


4
3
2
1
0
-1
1900
1950
2000
2050
year
stockindex (log)
consumption (log)

Puzzles

1. Equity Premium is high in the data
   - $\sigma'_{Rt}\sigma_{Ct}$ is empirically low for stocks, so γ must be very large (i.e., $\hat{\gamma} = 80$) to fit the high average return on stocks.
   - Issues
     - Large γ counterfactual with micro data
     - leads to the risk-free rate puzzle (risk-free rate too big).
2. Equity Premium is time varying in the data
   - $\gamma\sigma'_{Rt}\sigma_{Ct}$ must be time varying

Measuring Consumption

- In real life, consumption happens continuously between t and $t+1$, while returns corresponds to the price at a point in time.

Measuring Consumption

Table II

Alternative Measures of Consumption

Garbage excludes yard trimmings (see Table I). Durables, nondurables, and services are real per capita from NIPA. P–J is the 3-year future consumption expenditure growth as in Parker and Julliard (2005). Q4–Q4 is the fourth-quarter year-over-year consumption expenditure growth as in Jagannathan and Wang (2007). Panel A: R^M is the excess market return. Bootstrapped standard errors are from 10^6 simulations using blocks of size three. The last rows report pairwise correlations. Panel B: Standard errors are Newey–West with three lags.

Panel A: Sample Moments

	Garbage	Durables	Nondurables	Services	Nondur. & Serv.	P–J	Q4–Q4
Mean	1.47	4.62	1.67	2.55	2.21	4.96	2.21
	(0.36)	(0.91)	(0.23)	(0.24)	(0.21)	(0.60)	(0.22)
St. dev.	2.88	5.56	1.45	1.18	1.14	2.99	1.29
	(0.39)	(0.60)	(0.19)	(0.09)	(0.11)	(0.33)	(0.14)
Autocorr.	−14.51	28.90	22.09	51.58	40.01	67.72	32.78
	(11.54)	(11.49)	(12.23)	(11.09)	(10.84)	(6.01)	(11.16)
Corr. R^M	57.94	46.33	47.35	21.89	37.83	13.79	26.42
	(11.25)	(12.00)	(11.58)	(12.11)	(11.64)	(10.53)	(11.47)
Cov. R^M	26.86	41.47	11.04	4.15	6.92	6.64	5.49
	(10.32)	(14.32)	(4.27)	(2.48)	(2.70)	(4.84)	(2.71)
Garbage		42	51	45	53	13	36
Durables			78	57	74	61	65
Nondurables				57	85	54	72
Services					92	42	82
Nondur. & Serv.						53	87
P–J							61

Table I
Sensitivity of Stockholder, Top Stockholder, and Nonstockholder Consumption Growth to Aggregate Consumption Growth Across Horizons

The sensitivity of stockholder, top stockholder, and nonstockholder consumption growth to aggregate consumption growth from NIPA is reported over horizons of $S = 1, 2, 4, 8, 12, 16, 20$, and 24 quarters. The sensitivity of each group's consumption growth to aggregate consumption growth is computed as the regression coefficient from regressing a group's discounted consumption growth over horizon S on aggregate discounted consumption growth over the same horizon. Standard errors (in parentheses) on the regression sensitivity measure are computed using a Newey–West estimator that allows for autocorrelation of up to $S \times 3 - 1$ month lags. Group consumption growth rates are calculated using data from the Consumer Expenditure Survey over the period March 1982 to November 2004.

$S =$	1	2	4	8	12	16	20	24
Stockholder	0.68	0.93	1.21	1.57	2.12	2.68	2.68	2.42
(s.e.)	(0.35)	(0.37)	(0.32)	(0.36)	(0.39)	(0.49)	(0.49)	(0.41)
Top stockholder	0.70	1.01	1.56	2.14	2.88	3.94	3.91	3.48
(s.e.)	(0.90)	(0.77)	(0.62)	(0.49)	(0.53)	(0.67)	(0.73)	(0.63)
Nonstockholder	0.51	0.41	0.59	0.84	0.96	1.01	0.95	0.79
(s.e.)	(0.23)	(0.27)	(0.26)	(0.27)	(0.29)	(0.26)	(0.24)	(0.27)

Figure 2: Long-Run Stockholder Consumption Risk and Asset Returns (2009) JF

They obtain $\gamma \approx 10$ for the wealthiest third of stockholders with the largest holdings of equity