Heterogeneous Agent Models

Matthieu Gomez

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Model

Set-up

- Aggregate Endowment follows geometric Brownian motion

$$\frac{dY_t}{Y_t} = \mu dt + \sigma dW_t$$

- Assume two types of agents $i \in \{A, B\}$ with EZ utility and $\gamma_A < \gamma_B$
- Assume that agents of type A transition to type B with transition rate τ

- Denote C_i consumption of agent of type $i \in \{A, B\}$:

$$\frac{dC_i}{C_i} = \mu_{C_i} dt + \sigma_{C_i} dW_t$$

Denote V_i the wealth to consumption ratio of agents in group i

- Euler equation for agent $i \in \{A, B\}$ gives

$$\sigma_{Ci} = \frac{\kappa_t}{\gamma_i} \tag{1}$$

$$\mu_{Ci} = \psi_i (r_t - \rho) + \frac{1 + \frac{1}{\gamma_i}}{2\gamma_i} \kappa_t^2 \tag{2}$$

- Moreover, we have

$$\frac{1}{V_i} + \mu_{C_i} + \mu_{V_i} + \sigma_{C_i} \sigma_{V_i} = r + \kappa(\sigma_{C_i} + \sigma_{V_i})$$

Market clearing

- Total consumption is

$$C_t = \int_{i \in \mathbb{I}_A} C_{it} + \int_{i \in \mathbb{I}_B} C_{it}$$

- Denote x_t the share of consumptions by agents in group A,

$$x_t = \int_{i \in \mathbb{I}_A} C_{it} / \int_{i \in \mathbb{I}} C_{it}$$

- We have

$$\frac{dC_t}{C_t} = x_t \frac{dC_{At}}{C_{At}} + (1 - x_t) \frac{dC_B}{C_B} + x_t \tau (\frac{V_A}{V_B} - 1) dt$$

- Therefore,

$$\sigma = x \sigma_{C_A} + (1 - x) \sigma_{C_B} \tag{3}$$

$$\mu = x \mu_{C_A} + (1 - x) \mu_{C_B} + x \tau (\frac{V_A}{V_B} - 1) dt \tag{4}$$

Prices

- By plugging FOC Equation (1) into market clearing Equation (3), we obtain

$$\kappa_t = \frac{1}{\frac{x_t}{\gamma_A} + \frac{1-x_t}{\gamma_B}} \sigma \tag{5}$$

- By plugging FOC Equation (4) into market clearing Equation (2), we obtain

$$r = \rho + \frac{1}{x_t \frac{1}{\gamma_A} + (1 - x_t) \frac{1}{\gamma_B}} \left(\mu - \sum_i x_i \frac{1 + \frac{1}{\gamma_i}}{2\gamma_i} \kappa_t^2 - x_t \tau \left(\frac{V_A}{V_B} - 1 \right) \right) \tag{6}$$

- Based on the expression for prices, we can guess that x_t will be the (endogeneous) state variable of the economy
- We can derive the law of motion of $x_t = \int_{i \in \mathbb{I}_A} C_{it} / \int_{i \in \mathbb{I}} C_{it}$ using Ito's lemma:

$$\frac{dx_t}{x_t} = \frac{dC_{At}}{C_{At}} - \frac{dC_t}{C_t} + (\sigma^2 - \sigma_{CA}\sigma - \tau)dt$$

- Define μ_x and σ_x the arithmetic drift and volatility of x

$$dx = \mu_x dt + \sigma_x dW_t$$

We have

$$\sigma_x = x(\sigma_{CA} - \sigma) \tag{7}$$

$$\mu_x = x(\mu_{CA} - \mu + \sigma^2 - \sigma_{CA}\sigma - \tau) \tag{8}$$

Solution Method

1. Given the function V_A, V_B:
2. Inject expression for κ in Euler equation for σ_C to express σ_{CA}

$$\sigma_{C_A} = \frac{\sum \frac{x_j}{\gamma_j}}{\gamma_A} \sigma$$

3. Use Ito's lemma and expression for σ_x to express σ_x in term of V_A, V_B

$$\sigma_x = \frac{x(1-x) \sum \frac{x_j}{\gamma_j}}{\gamma_A \gamma_B} (\gamma_A - \gamma_B) \sigma$$

4. Given σ_x, we know r, μ_x
5. Therefore, using Ito's lemma, we get $\sigma_{V_A}, \sigma_{V_B}, \mu_{V_A}, \mu_{V_B}$, and therefore we can write market pricing for two agents as a system of two ODES.

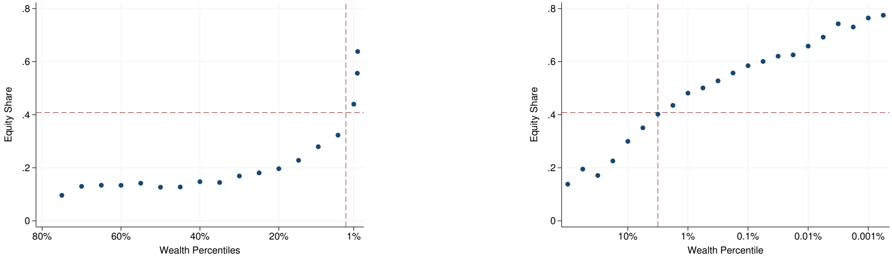
Distribuiton

- One can show that the wealth distribution has a thick right tail

$$\zeta = \frac{\tau}{\mu_{WA} - \frac{1}{2}\sigma_{WA}^2 - (\mu - \frac{1}{2}\sigma^2)}$$

Data

Figure 1: Equity Share = $\frac{\text{Public Equity + Private Equity}}{\text{Wealth}}$



Data source: Survey of Consumer Finance (SCF), a cross sectional survey of US households from 1989 to 2013. Horizontal line represents the average equity share. Vertical line splits the sample in two of same aggregate wealth.

Fact 2: When Stock Returns are High, Wealth Inequality Increases $\sigma_A - \sigma > 0$

$$\log \frac{\text{Top Wealth}_{t+1}}{\text{Top Wealth}_{t-1}} - r_t^f = \alpha + \beta \times (r_t^M - r_t^f) + \gamma \times r_t^f + \epsilon_t$$

	Groups of Households Defined by Wealth Percentiles				
	Flow of Funds	Estate Tax Returns			Forbes
	All Households	1 − 0.1%	0.1 − 0.01%	Top 0.01%	Top 100
	(1)	(2)	(3)	(4)	(5)
Excess Stock Returns	0.44***	0.52***	0.66***	0.75***	0.71***
	(0.13)	(0.18)	(0.17)	(0.23)	(0.18)
R^2	0.49	0.45	0.58	0.40	0.34
Period	1917-1999	1917-1999	1917-1999	1917-1999	1983-2014
N	54	54	54	54	32

Estimation via OLS. Standard errors in parentheses and estimated using Newey-West with 3 lags. *,** ,*** indicate significance at the 0.1, 0.05, 0.01 levels.

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	All Households	1 — 0.1%	0.1 — 0.01%	Top 0.01%	Top 100
	(1)	(2)	(3)	(4)	(5)
Excess Stock Returns		0.09*	0.22***	0.31**	0.27*
		(0.05)	(0.07)	(0.14)	(0.19)
R^2		0.20	0.33	0.14	0.18
Period		1917-1999	1917-1999	1917-1999	1983-2013
N		54	54	54	31

Estimation via OLS. Standard errors in parentheses and estimated using Newey-West with 3 lags. $^{*}, ^{**}, ^{***}$ indicate significance at the 0.1, 0.05, 0.01 levels.

Fact 3: High Wealth Inequality Predicts Lower Future Excess Returns $\partial\kappa/\partial x < 0$

Figure 2: The Wealth Share of the Top 0.01% and Average Excess Returns