Distribution

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Power Law Distribution

Power Law

- X follows power law distribution if wealth density g follows

$$g(x) = Cx^{-\zeta-1} \text{ for } x \geq x_{\min}$$

  - $\zeta > 0$ is called the power law exponent. When $\zeta = 1$, it is called Zipf's law
  - Note that all moments higher than ζ are infinite
  - C is pinpointed by the fact density sums to one, which gives

$$g(x) = \frac{\zeta}{x_{\min}}\left(\frac{x}{x_{\min}}\right)^{-\zeta-1}$$

- X has a thick right tail if wealth density g follows

$$g(x) \sim Cx^{-\zeta-1} \text{ for } x \to +\infty$$

Power Law

- For a distribution with a thick right tail, we have

$$\begin{aligned} P(X \geq x) &\sim \int_x^{+\infty} Cu^{-\zeta-1} du \\ &\sim x^{-\zeta} \end{aligned}$$

- This gives a natural way to test for power law distribution and measure ζ: plot $\log P(X \geq x)$ in term of $\log x$

How to Obtain Power Law?

- Suppose that wealth follows geometric Brownian Motion

$$\frac{dx}{x} = \mu dt + \nu dZ_{it}$$

In log, using Ito's lemma,

$$d \ln x = (\mu - \frac{\nu^2}{2})dt + \nu dZ_{it}$$

- This means that $\ln x_t \sim N((\mu - \frac{\nu^2}{2})t, \nu^2 t)$

How to Obtain Power Law?

First way to have stationary distribution: assume that the process is reflected if wealth hits some lower bound $x_{\min}$.

- Kolmogorov Forward is

$$0 = -\mu x_{\min} g(x_{\min}) + \frac{1}{2}\partial_x(\nu^2 x_{\min}^2 g(x_{\min}))$$
$$0 = -\partial_x(\mu x g(x)) + \frac{1}{2}\partial_{xx}(\nu^2 x^2 g(x)) \text{ for } x > x_{\min}$$

- Guess

$$g(x) = Cx^{-\zeta-1} \text{ for } x \geq x_{\min}$$

This satisfies Kolmogorov Forward if

$$0 = \zeta\mu + \frac{\zeta(\zeta-1)}{2}\nu^2$$
$$\Rightarrow \zeta = 1 - \frac{2\mu}{\nu^2}$$

The density existsiff $\zeta > 0$, that is, $\mu - \nu^2/2 < 0$

How to Obtain Power Law?

Second way to have stationary distribution: assume that people die with rate δ and are born with initial distribution ψ that has thin tail (i.e., $\psi(x)x^n \to 0$ for any n as $x \to +\infty$)

- Kolmogorov Forward is

$$0 = -\partial_x(\mu x g(x)) + \frac{1}{2}\partial_{xx}(\nu^2 x^2 g(x)) + \delta(\psi(x) - g(x))$$

- Guess

$$g(x) \sim Cx^{-\zeta-1} \text{ as } x \geq x_{\text{min}}$$

We obtain

$$0 = \zeta\mu + \frac{\zeta(\zeta-1)}{2}\nu^2 - \delta$$

There is always a positive root given by

$$\zeta = \frac{1 - \frac{2\mu}{\nu^2} + \sqrt{(1 - \frac{2\mu}{\nu^2})^2 + 8\frac{\delta}{\nu^2}}}{2}$$

Law of Motion of Top Shares

Law of Motion of Top Wealth Shares

- The top quantile q_t is defined as

$$p = \int_{q_t}^{+\infty} g_t(x)dx$$

During a short period of time dt:

$$0 = \int_{q_t}^{+\infty} dg_t(x)dx - g_t(q_t)dq_t$$

$$\Rightarrow dq_t = \frac{\int_{q_t}^{+\infty} dg_t(x)dx}{g_t(q_t)}$$

- The top wealth share is defined as

$$S_t = \int_{q_t}^{+\infty} xg_t(x)dx$$

During a short period of time dt:

$$\begin{aligned} dS_t &= \int_{q_t}^{+\infty} xdg_t(x)dx - q_t g_t(q_t)dq_t \\ &= \int_{q_t}^{+\infty} xdg_t(x)dx - \int_{q_t}^{+\infty} dg_t(x)dx \end{aligned}$$

$$\Rightarrow dS_t = \int_{q_t}^{+\infty} (x - q_t)dg_t(x)dx$$

Diffusion Process

- Assume that wealth of households relative to the average wealth in the economy follows a geometric Brownian Motion:

$$\frac{dx_{it}}{x_{it}} = \mu_t(x)dt + \nu_t(x)dZ_{it}$$

- Then we have

$$\begin{aligned} dS_t &= \int_{q_t}^{+\infty} (x - q_t)dg_t(x)dx \\ &= \int_{q_t}^{+\infty} (x - q_t)(-\partial_x(\mu_t(x)xg_t(x)) + \frac{1}{2}\partial_{xx}(^2(x)x^2 g_t(x)))dx \end{aligned}$$

Integrating by parts, we obtain

$$\frac{dS_t}{S_t} = E^x[\mu_t(x)|x \geq q_t]dt + \frac{1}{2}\frac{g_t(q_t)q_t^2}{S_t}\nu_t(q_t)^2 dt$$

  - q_t denotes the wealth threshod at percentile p
  - $g_t(q_t)$ denotes the density of wealth at percentile p

Heuristic Proof

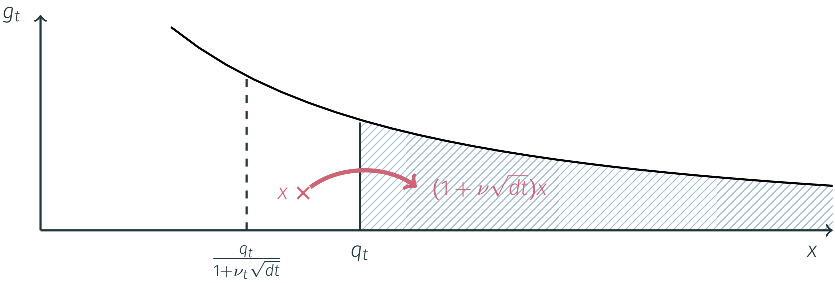
- Alternatively, we can prove the law of motion of top wealth shares heuristically. Take the case $\mu = 0$ to simplify. Let us show that the the entry and exit in top wealth shares sum up to $\frac{g_t(q_t)q_t^2}{2S_t}\nu_t^2 dt$
- The process

$$\frac{dw_t}{w_t} = \nu_t dZ_t$$

can be seen as limit of the process

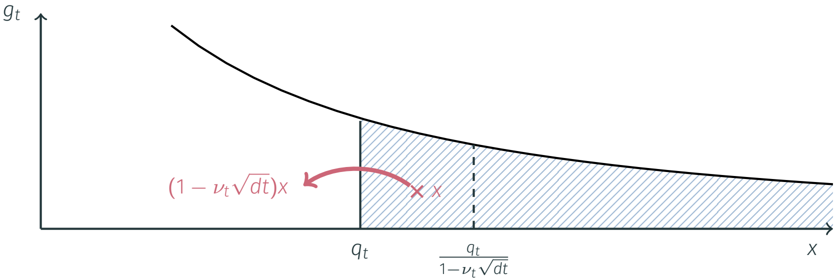
$$\begin{aligned} w_{t+\Delta t} &= (1+\nu_t\sqrt{\Delta t})w_t \text{ with probability } 1/2 \\ &= (1-\nu_t\sqrt{\Delta t})w_t \text{ with probability } 1/2 \end{aligned}$$

Heuristic Proof



$$dr_{\text{entry}} \approx \int_{\frac{q_t}{(1+\nu_t\sqrt{dt})}}^{q_t} \frac{(1+\nu_t\sqrt{dt})x - q_t}{2S_t} g_t(x)dx \approx \frac{g_t(q_t)q_t^2}{4S_t}\nu_t^2 dt$$

Heuristic Proof



$$dr_{\text{exit}} \approx \int_{q_t}^{\frac{q_t}{(1-\nu_t\sqrt{dt})}} \frac{q_t - (1-\nu_t\sqrt{dt})x}{2S_t} g_t(x)dx \approx \frac{g_t(q_t)q_t^2}{4S_t}\nu_t^2 dt$$

Pareto Distribution

- Assume that the wealth distribution is Pareto at time t

$$\mathbb{P}_t(x_{it} \geq x) = Cx^{-\zeta}$$

Then we have

$$\frac{g_t(x)x^2}{\int_x u g_t(u) du} = \zeta - 1$$

- The growth of top wealth share S_t is:

$$\frac{dS_t}{S_t} = E^x[\mu_t(x) | x \geq q_t] dt + \frac{\zeta - 1}{2} \nu_t(q_t)^2 dt$$

  - The role of idiosyncratic volatility is the same for all percentiles in the right tail
  - As $\zeta \to 1$ (Zipf's law), role of idiosyncratic volatility is null
  - The distribution is stationary iff $dS_t = 0$, i.e.

$$0 = E^x[\mu_t(x) | x \geq q_t] + \frac{\zeta - 1}{2} \nu_t(q_t)^2$$

  This gives the equation for the tail index ζ derived earlier

Mapping Decomposition to the Data

Accounting Decomposition

- Denote x_{it} the wealth of household i relative to total U.S. wealth
- For a top percentile p, denote top wealth share S_t:

$$S_t = \sum_{i \in \mathcal{T}_t} x_{it}$$

where $\mathcal{T}_t$ denotes the set of households in the top percentile at time t.

Accounting Decomposition

- The growth of S_t between t and $t+1$ can be written as:

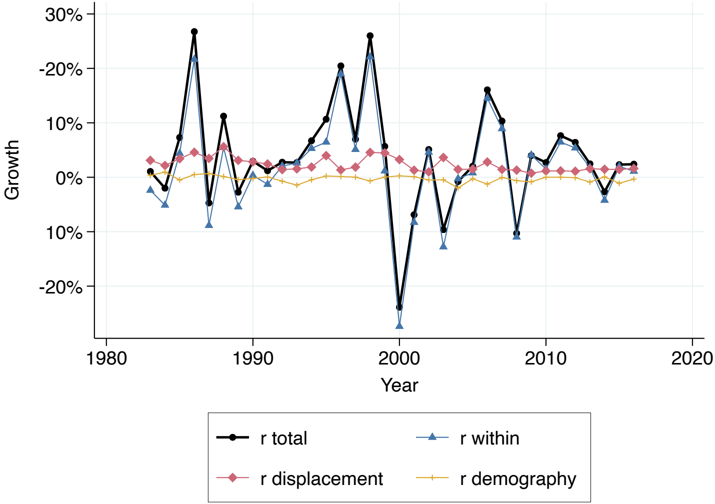
$$\frac{S_{t+1}}{S_t} - 1 = \frac{\sum_{i \in \mathcal{T}_{t+1}} x_{it+1}}{\sum_{i \in \mathcal{T}_t} x_{it}} - 1$$

$$= \frac{\sum_{i \in \mathcal{T}_t} x_{it+1}}{\sum_{i \in \mathcal{T}_t} x_{it}} - 1 + \frac{\sum_{i \in \mathcal{E}} x_{it+1} - \sum_{i \in \mathcal{X}} x_{it+1}}{\sum_{i \in \mathcal{T}_t} x_{it}}$$

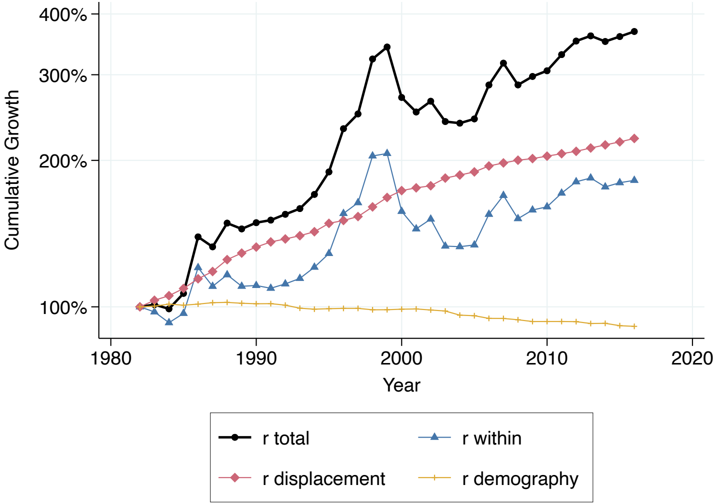
  - $\mathcal{E}$ denotes the set of households that *enter* the top between t and $t+1$
  - $\mathcal{X}$ denotes the set of households that *exit* the top between t and $t+1$

▸ Demographics ▸ Measurement Error

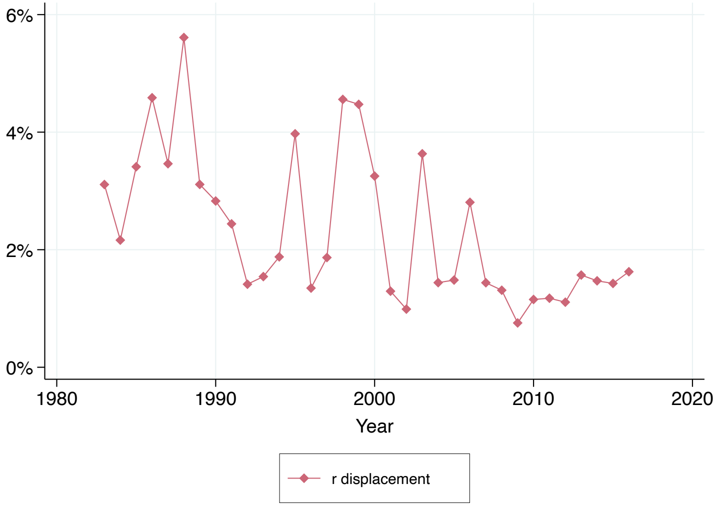
Yearly Decomposition



Yearly Decomposition



What Drives Displacement?



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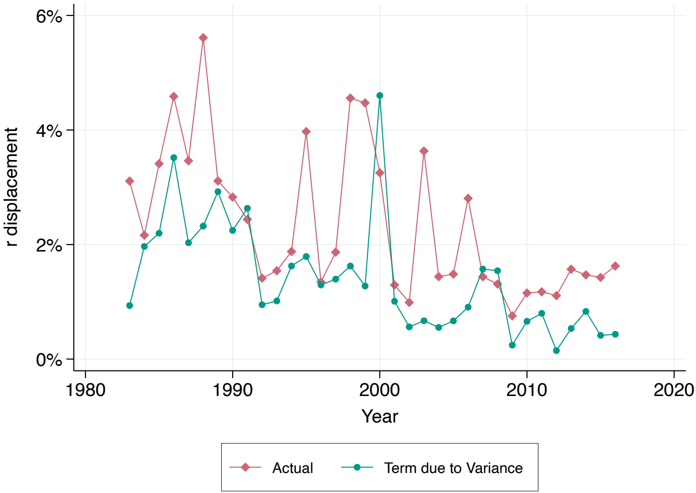
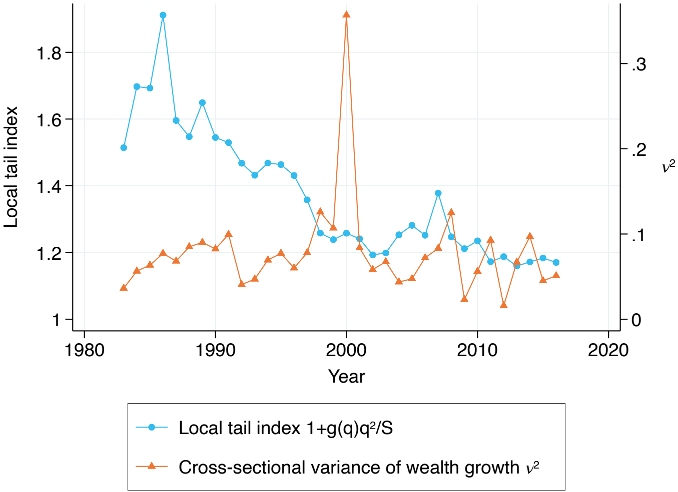


Term due to Variance is $\frac{1}{2}\frac{g_t(q_t)q_t^2}{S_t}\nu_t^2(q_t)$.

What Drives Displacement?



Displacement Within v.s. Between Industries

$$\frac{1}{2}\frac{g_t(q_t)q_t^2}{S_t}\nu^2 = \frac{1}{2}\frac{g_t(q_t)q_t^2}{S_t}\nu_{\text{within}}^2 + \frac{1}{2}\frac{g_t(q_t)q_t^2}{S_t}\nu_{\text{between}}^2$$