Disaster Risk

Matthieu Gomez

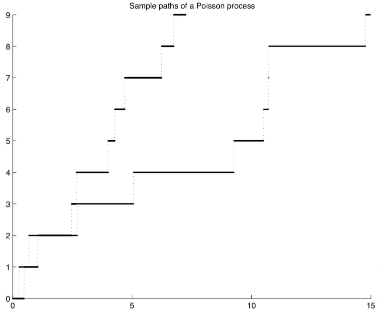
November 21, 2019

Jump Process

Poisson Process

- A Poisson Process with intensity λ is defined as

$$N_{t+\Delta t} = \begin{cases} N_t + 1 \text{ with probability } \lambda \Delta t \\ N_t \text{ with probability } 1 - \lambda \Delta t \end{cases}$$



Compound Poisson Process

- A *Compound* Poisson Process with intensity λ is defined as

$$Y_{t+\Delta t} = \begin{cases} Y_t + J_t \text{ with probability } \lambda \Delta t \\ Y_t \text{ with probability } 1 - \lambda \Delta t \end{cases}$$

  J_t, the jump size, is itself a random variable
- We write

$$dY_t = J_t dN_t$$

- Note that

$$E[dY_t] = E[J_t] \lambda dt$$

Ito Lemma

- Consider the process x_t

$$dx_t = \mu_t dt + \sigma_t dZ_t + J_t dN_t$$

- Ito's Lemma with jumps. For a twice differentiable function f, we have

$$df(x_t) = \underbrace{f'(x_{t\text{-}})(\mu_t dt + \sigma_t dZ_t) + \frac{1}{2} f''(x_{t\text{-}})\sigma_t^2 dt}_{\text{Diffusive Part}} + \underbrace{(f(x_{t\text{-}} + J_t) - f(x_{t\text{-}}))dN_t}_{\text{Jump Part}}$$

In particular

$$E[df(x_t)] = f'(x_{t\text{-}})\mu_t dt + \frac{1}{2} f''(x_{t\text{-}})\sigma_t^2 dt + E[f(x_{t\text{-}} + J_t) - f(x_{t\text{-}})]\lambda dt$$

Ito Lemma: Example

- Consider the process x_t

$$dx_t = \mu_t dt + \sigma_t dZ_t + J_t dN_t$$

- Denote $y_t = e^{x_t}$. Applying Ito's lemma, we have

$$\frac{dy_t}{y_{t\text{-}}} = \left(\mu_t + \frac{\sigma_t^2}{2}\right) dt + \sigma_t dZ_t + (e^{J_t} - 1) dN_t$$

Cumulant Generating Function

- For a random variable U, the Cumulant Generating Function (CGF) is defined as

$$\mathsf{c}(\theta) = \log E\left[e^{\theta U}\right]$$

- Cumulant of order k is defined as defined as $\partial_\theta^k \mathsf{c}(U)|_{\theta=0}$. Using Taylor expansion, we can write:

$$\mathsf{c}(\theta) = \sum_{k\geq 1} \frac{\kappa_k}{k!}\theta^k$$

In particular,

$$\mathsf{c}(\theta) = \text{mean} \cdot \theta + \frac{1}{2} \cdot \text{sd}^2 \cdot \theta^2 + \frac{\text{skewness}}{3!} \cdot \text{sd}^3 \cdot \theta^3 + \frac{\text{excess kurtosis}}{4!} \cdot \text{sd}^4 \cdot \theta^4 + \dots$$

- The normal distribution is the only distribution such that $\kappa_k = 0$ for $k > 2$.

Cumulant Generating Function

- Similarly, we can define the instantaneous CGF of a process x_t as:

$$\mathrm{c}_t(\theta)dt = \frac{\partial \log E_t\left[e^{\theta(x_{t+\Delta}-x_{t\text{-}})}\right]}{\partial \Delta}|_{\Delta=0}$$
$$= E_t\left[\frac{de^{\theta x_t}}{e^{\theta x_{t\text{-}}}}\right]$$

- For instance, for a process x_t

$$dx_t = \mu_t dt + \sigma_t dZ_t + J_t dN_t$$

We have

$$\mathrm{c}_t(\theta) = \theta\mu_t + \frac{1}{2}\theta^2\sigma_t^2 + \lambda_t(E_t[e^{\theta J_t}] - 1)$$

- Similarly, we can define instantaenous cumulants:

$$\kappa_{1t} = \mu_t + \lambda_t E_t[J]$$
$$\kappa_{2t} = \sigma_t^2 + \lambda_t E_t[J^2]$$
$$\kappa_{kt} = \lambda_t E_t[J^k] \text{ for } k > 2$$

Portfolio Problem with Jumps

Consumption process.

Assume the following process for aggregate consumption:

$$\frac{dC_t}{C_{t\text{-}}} = \mu_C dt + \sigma_C dZ_t + (e^{J_t} - 1)dN_t$$

where

1. dZ_t is a standard Brownian motion (potentially multi-dimensional)
2. dN_t is a Poisson process with intensity λ_t, which follows a diffusion:

$$d\lambda_t = \mu_\lambda(\lambda_t)dt + \sigma_\lambda(\lambda_t)dZ_t$$

Portfolio Problem

- Denote R_t the value of the consumption claim and guess

$$\frac{dR_t}{R_{t\text{-}}} = \mu_{Rt} dt + \sigma_{Rt} dZ_t + (e^{J_t} - 1) dN_t$$

Note that in particular

$$E[\frac{dR_t}{R_{t\text{-}}}] = \mu_{Rt} dt + E[e^{J_t} - 1] \lambda_t dt$$

- Denote U the utility function. The portfolio problem is

$$\rho \mathcal{U}_t = \max_{C,\alpha} \{ U(C_t) + E[d\mathcal{U}_t] \}$$

$$\frac{dW_t}{W_{t\text{-}}} = \underbrace{(r_t + \alpha_t(\mu_{Rt} - r_t) - \frac{C_t}{W_t})}_{\mu_{Wt}} dt + \alpha_t \sigma_{Rt} dZ_t + \alpha_t (e^{J_t} - 1) dN_t$$

r_t is the risk free rate, C_t is consumption, α is share of wealth invested in the risky asset

Portfolio Problem

- Guess there exists a process V_t such that

$$\mathcal{U}_t(W) = \frac{W^{1-\gamma}}{1-\gamma} V_t^{\gamma}$$

- Guess that V_t is a diffusion process, i.e.

$$\frac{dV_t}{V_t} = \mu_{Vt} dt + \sigma_{Vt} dZ_t$$

  Because jumps change the level of consumption, but not the dynamics of consumption going forward, we can guess that V is not impacted by jumps (when there is a jump, both C and W jump by the same magnitude).

FOC Portfolio Problem

- HJB is

$$0 = \max_{C_t,\alpha_t}\Big\{\rho + (1-\gamma)\mu_{Wt} - \frac{\gamma(1-\gamma)}{2}\alpha_t^2\sigma_{Rt}^2 + +\lambda_t E\left[(1+\alpha_t(e^{J_t}-1))^{1-\gamma} - 1\right]$$
$$+\gamma\mu_{Vt} + 2\gamma(\gamma-1)\sigma_{Vt}^2 + \gamma(1-\gamma)\alpha_t\sigma_{Rt}\sigma_{Vt}\Big\}$$

- FOCs give

$$\frac{C_t}{W_t} = \frac{1}{V_t}$$
$$0 = \mu_{Rt} - r_t - \gamma\alpha\sigma_{Rt}^2 + \frac{1-\gamma}{\psi-1}\alpha_t\sigma_{Rt}\sigma_{Vt} + \lambda_t E\left[(e^{J_t}-1)(1+\alpha(e^{J_t}-1))^{-\gamma}\right]$$

- With Market Clearing for α_t, FOC becomes

$$0 = \mu_{Rt} - r_t - \gamma\sigma_{Rt}^2 + \gamma\sigma_{Rt}\sigma_{Vt} + \lambda_t E\left[(e^{J_t}-1)e^{-\gamma J_t}\right]$$

Solution of Portfolio Problem

- Define

$$\mathbf{m}(\theta) = E[e^{\theta J_t} - 1]$$

- After some manipulations (see previous slides), we obtain the following system for risk premium, risk free rate, and PDE for V

$$E\left[\frac{dR_t}{R_{t\text{-}}}\right] - r_t = \gamma\sigma_C^2 + \lambda_t \mathbf{m}(1) + \lambda_t \mathbf{m}(-\gamma) - \lambda_t \mathbf{m}(1-\gamma)$$

$$r_t = \rho + \gamma\mu_C - \frac{1+\gamma}{2}\gamma\sigma_C^2 - \lambda_t \mathbf{m}(-\gamma)$$

$$0 = \frac{1}{V_t} + \mu_{Vt} + \mu_C + \sigma_C\sigma_{Vt} + \lambda_t \mathbf{m}(1) - E\left[\frac{dR_t}{R_{t\text{-}}}\right]$$

As usual, this system of three equations can be written (and solved) as an ODE for $V(\lambda)$

Case of Epstein Zin

- Setup from Wachter "Can Time-Varying Risk of Rare Disasters Explain Aggregate Stock Market Volatility?"
- The system of equations becomes

$$E_t\left[\frac{dR_t}{R_{t\text{-}}}\right] - r_t = \underbrace{\gamma\sigma_C^2}_{\text{standard model}} + \underbrace{\lambda_t \mathrm{m}(1) + \lambda_t \mathrm{m}(-\gamma) - \lambda_t \mathrm{m}(1-\gamma)}_{\text{static disaster}} + \underbrace{\frac{\gamma\psi - 1}{\psi - 1}\sigma_{Vt}^2}_{\text{time-varying intensity}}$$

$$r_t = \underbrace{\rho + \frac{\mu_C}{\psi} - \frac{1+\frac{1}{\psi}}{2}\gamma\sigma_C^2}_{\text{standard model}} \underbrace{-\lambda_t \mathrm{m}(-\gamma) + \frac{\frac{1}{\psi}-\gamma}{1-\gamma}\lambda_t \mathrm{m}(1-\gamma)}_{\text{static disaster}} - \underbrace{\frac{\gamma\psi - 1}{2(\psi - 1)}\sigma_{Vt}^2}_{\text{time varying intensity}}$$

$$0 = \frac{1}{V_t} + \mu_{Vt} + \mu_C + \lambda_t \mathrm{m}(1) - E_t\left[\frac{dR_t}{R_{t\text{-}}}\right]$$

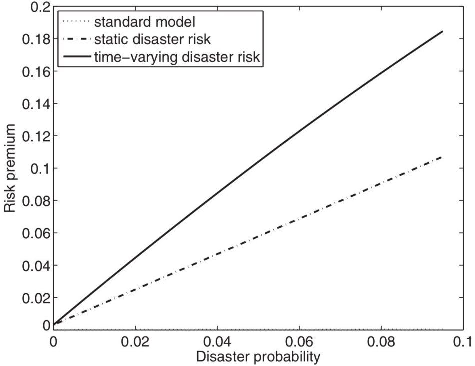


Figure 3. Decomposition of the equity premium in the time-varying disaster risk model. The solid line shows the instantaneous equity premium (the expected excess return on equity less the expected return on the government note), the dashed line shows the equity premium in a static model with disaster risk, and the dotted line shows what the equity premium would be if disaster risk were zero.

Empirical Test

Empirical Support: Barro (2006), Barro Ursua (2009)

- Are there disasters? Barro (2006)
- Do prices decline when probability of disaster increase? Barro and Ursua (2009)
  1. P(Consumption Drops by 10% | No Drop in Asset prices) = 1%
  2. P(Consumption Drops by 10% | Drop in Asset prices) = 3.8%

Critiques: Consumption

Two critiques about consumption model vs data

$$\frac{dC_t}{C_{t\text{-}}} = \mu dt + \sigma dZ_t^C + (e^{J_t} - 1)dN_t$$

1. In the data, disasters occur across a few years, not instantaneously. Barro 2006 measures disaster from peak to through drop but maybe we should only use annual (monthly?) maximum drop in consumption.
2. In the data, there is usually a a rapid recovery of consumption after disaster, i.e. impact of disaster on consumption is transitory, not permanent

Incorporating these two facts in the model would make the price of disaster much smaller

Critiques: Price of Disaster Insurance

- By buying stock market + deep below-the-money- index put option, one can hold an asset that mimicks the stock market return, but never decreases by more than 15% in a month.
- Welch (2016) shows that this porftolio almost earns the same premium as the stock market. This suggests that the market price of disaster is fairs small