Continuous Time

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Continuous Time Mathematics

Standard Brownian Motion

- A standard brownian motion Z_t is the unique continuous time process with the following properties
  1. Z_t is continuous almost surely
  2. Z_t has stationary, independent increments
  3. For any $s > 0$,

$$Z_{t+s} - Z_t \sim N(0, s)$$

- Heuristically, the standard brownian motion can be seen as the limit of the discrete time process

$$Z_{t+\Delta t} = Z_t + \sqrt{\Delta t} \times \epsilon$$
$$\text{where } \epsilon \sim N(0, 1)$$

as $\Delta t \to 0$

Markovian Diffusion Process

- A Markovian Diffusion process is a process defined as:

$$dx_t = \mu(x_t)dt + \sigma(x_t)dZ_t$$

  where Z_t is the standard brownian motion defined in previous line.
- $\mu(x_t)$ is called its instantaneous drift and $\sigma(x_t)$ is called its instantaneous volatility. We can write:

$$E_t[dx_t] = \mu(x_t)dt$$
$$Var_t[dx_t] = \sigma(x_t)^2 dt$$

- Heuristically, it can be seen as the limit of the discrete time process

$$x_{t+\Delta t} = x_t + \mu(x_t)\Delta t + \sigma(x_t)\sqrt{\Delta t} \times \epsilon$$
$$\text{where } \epsilon \sim N(0,1)$$

  as $\Delta t \to 0$

Examples

- An **arithmetic** brownian motion is a process $x_t, t \in [0, \infty)$ defined as:

$$dx_t = \mu dt + \sigma dZ_t$$

  where Z_t is a standard brownian motion

- A **geometric** brownian motion is a process $x_t, t \in [0, \infty)$ defined as:

$$dx_t = \mu x_t dt + \sigma x_t dZ_t$$

  where Z_t is a standard brownian motion

Examples

- A Ornstein-Uhlenbeck is a process $x_t, t \in [0, \infty)$ defined as:

$$dx_t = -\kappa(x_t - \bar{x})dt + \sigma dZ_t$$

  with $\kappa > 0$

- It can be seen as the continuous-time equivalent of an AR(1) process. To see this, note that, for small Δt:

$$\begin{aligned} x_{t+\Delta t} &= x_t - \kappa(x_t - \bar{x})\Delta t + \sigma\sqrt{\Delta t} \times \epsilon \\ &= \kappa\bar{x}\Delta t + (1 - \kappa\Delta t)x_t + \sigma\sqrt{\Delta t} \times \epsilon \end{aligned}$$

  which has indeed the form of an AR(1): $x_{t+\Delta t} = c + \phi x_t + e_{t+\Delta t}$

Ito's lemma

- Take a Markovian diffusion process

$$dx_t = \mu(x_t)dt + \sigma(x_t)dZ_t$$

- Ito's lemma says that, for any smooth function f, we have:

$$df(x_t) = \left(f'(x_t)\mu(x_t) + \frac{1}{2}f''(x_t)\sigma^2(x_t)dt\right) dt + f'(x_t)\sigma(x_t)dZ_t$$

  Alternatively, it can be written as:

$$df(x_t) = \underbrace{f'(x_t)dx_t}_{\text{linear term}} + \underbrace{\frac{1}{2}f''(x_t)\sigma^2(x_t)dt}_{\text{"Ito's term"}}$$

Ito's Lemma

Heuristic derivation of Ito's Lemma:

- Start with Taylor expansion

$$f(x_{t+\Delta t}) - f(x_t) \approx f'(x_t)(x_{t+\Delta t} - x_t) + \frac{1}{2} f''(x_t)(x_{t+\Delta t} - x_t)^2 + \dots$$

- Taking the expectation, we have

$$\begin{aligned} E_t[f(x_{t+\Delta t}) - f(x_t)] &= f'(x_t)E[x_{t+\Delta t} - x_t] + \frac{1}{2} f''(x_t)\left(Var(x_{t+\Delta t} - x_t) + E[x_{t+\Delta t} - x_t]^2\right) + \dots \\ &= f'(x_t)\mu(x_t)\Delta t + \frac{1}{2} f''(x_t)\left(\sigma(x_t)^2\Delta t + (\mu(x_t)\Delta t)^2\right) + \dots \\ &= f'(x_t)\mu(x_t)\Delta t + \frac{1}{2} f''(x_t)\sigma(x_t)^2\Delta t + o(\Delta t) \\ \Rightarrow E_t[df(x_t)] &= f'(x_t)\mu(x_t)dt + \frac{1}{2} f''(x_t)\sigma(x_t)^2 dt \end{aligned}$$

  This gives the instantaneous drift of $f(x_t)$

- Taking the variance, we have

$$\begin{aligned} Var_t(f(x_{t+\Delta t}) - f(x_t)) &\approx f'(x_t)^2 Var(x_{t+\Delta t} - x_t) + o(\Delta t) \\ &= f'(x_t)^2\sigma(x_t)^2 dt + o(\Delta t) \\ \Rightarrow Var_t[df(x_t)] &= f'(x_t)\sigma(x_t)dt \end{aligned}$$

  This gives the instantaneous volatility of $f(x_t)$

Example

- Take a geometric Brownian Motion:

$$dx_t = \mu x_t dt + \sigma x_t dZ_t$$

- Show that Ito's lemma gives:

$$d \ln x_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dZ_t$$

  In other words, a geometric Brownian Motion with geometric drift μ and geometric volatility σ corresponds to the exponential of a Brownian Motion with drift $\mu - \frac{1}{2}\sigma^2$ and volatility σ

- Show that, for $\alpha \in \mathbb{R}$, Ito's lemma gives

$$\frac{dx^\alpha}{x^\alpha} = \alpha \frac{dx}{x} + \frac{1}{2}\alpha(\alpha - 1)\sigma^2 dt$$

Case of Multiple Variables

- Consider the following system

$$dx_1 = \mu_1(x_1, \ldots, x_n)dt + \sigma_1(x_1, \ldots, x_n)dZ_t$$
$$\vdots$$
$$dx_n = \mu_n(x_1, \ldots, x_n)dt + \sigma_n(x_1, \ldots, x_n)dZ_t$$

- It can be written in vector form:

$$d\boldsymbol{x}_t = \boldsymbol{\mu}(\boldsymbol{x})dt + \boldsymbol{\sigma}(x)dZ_t$$

where $\boldsymbol{x}$, $\boldsymbol{\mu}(\boldsymbol{x})$, and $\boldsymbol{\sigma}(\boldsymbol{x})$ are vectors of size $n \times 1$

- For any smooth function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, Ito's lemma gives:

$$df(\boldsymbol{x}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial f}{\partial x_i \partial x_j} \sigma_i \sigma_j dt$$

Example

- Denote

$$\frac{dx}{x} = \mu_x dt + \sigma_x dZ_t$$
$$\frac{dy}{y} = \mu_y dt + \sigma_y dZ_t$$

- Show that Ito's lemma gives:

$$\frac{d(xy)}{xy} = \frac{dx}{x} + \frac{dy}{y} + \sigma_x \sigma_y dt$$

- More generally, show that, for $\alpha, \beta \in \mathbb{R}$, Ito's lemma gives:

$$\frac{d(x^\alpha y^\beta)}{x^\alpha y^\beta} = \alpha \frac{dx}{x} + \beta \frac{dy}{y} + \frac{1}{2}\alpha(\alpha - 1)\sigma_x^2 dt + \frac{1}{2}\beta(\beta - 1)\sigma_y^2 dt + \alpha\beta\sigma_x\sigma_y dt$$

Case of Multiple Shocks

- Consider the following system

$$dx_t = \mu_1(x_t)dt + \sigma_1(x_t)dZ_1 + \cdots + \sigma_m(x_t)dZ_m$$

where $dZ_{1t}, \ldots dZ_{mt}$ are uncorrelated shocks

- It can be written in vector form:

$$dx_t = \mu(x_t)dt + \sigma(x_t)'dZ_t$$

where $\sigma(x_t), dZ_t$ are vectors of size $m \times 1$

- Consider a function $f : \mathbb{R} \longrightarrow \mathbb{R}$. Ito's lemma says:

$$df(x_t) = f'(x_t)dx_t + \frac{1}{2}f''(x_t)\sigma'(x_t)\sigma(x_t)dt$$

Kolmogorov Backward

Analytical Result

Theorem (Kolmogorov Backward)
Take a Markov diffusion process

$$dx_t = \mu(x_t)dt + \sigma(x_t)dZ_t$$

The function $u(x,t) = E[f(x_T)|x_t = x]$ *is the solution of the following PDE:*

$$u(x,T) = f(x)$$
$$0 = \partial_t u + \mu \partial_x u + \frac{1}{2}\sigma^2 \partial_{xx} u$$

Proof.

Consider a small time period Δt

$$\begin{aligned} u(x,t) &= E[f(x_T)|x_t = x] \\ &= E[E[f(x_T)|x_{t+\Delta t}]|x_t = x] \\ &= E[u(x_{t+\Delta t}, t+\Delta t)|x_t = x] \end{aligned}$$

Substracting by $u(x, t+\Delta t)$

$$\begin{aligned} u(x,t) - u(x,t+\Delta t) &= E[u(x_{t+\Delta t}, t+\Delta t) - u(x_t, t+\Delta t)|x_t = x] \\ \Rightarrow 0 &= \partial_t u + \mu \partial_x u + \frac{1}{2}\sigma^2 \partial_{xx} u \end{aligned}$$

□

Theorem (Kolmogorov Backward on a Bounded Space)
Take a Markov diffusion process

$$dx_t = \mu(x_t)dt + \sigma(x_t)dZ_t$$

with reflecting boundary at $\underline{x}, \overline{x}$.

The function $u(x,t) = E[f(x_T)|x_t = x]$ *is the solution of the following PDE:*

$$u(x,T) = f(x)$$
$$0 = \partial_t u + \mu \partial_x u + \frac{1}{2}\sigma^2 \partial_{xx} u$$
$$\partial_x u(\underline{x}, t) = 0$$
$$\partial_x u(\overline{x}, t) = 0$$

Solution Method

- How to solve for $u(x,t)$?
  1. Define a state space $\mathbf{x} = (x_1, x_2, \ldots, x_I)$ with a step Δx (uniform grid). Define a time step Δt
  2. Define $u_i^n = u(x_i, T - n\Delta t)$. One can solve the PDE using "finite-difference" method with upwinding. We have $u_i^0 = f(x_i)$ and, for $n > 0$, the recurrence relation:

$$\begin{cases} \frac{u_i^{n+1} - u_i^n}{\Delta t} = \mu_i^+ \frac{u_{i+1}^n - u_i^n}{\Delta x} + \mu_i^- \frac{u_i^n - u_{i-1}^n}{\Delta x} + \frac{1}{2}\sigma_i^2 \frac{u_{i+1}^n + u_{i-1}^n - 2u_i^n}{(\Delta x)^2} & \text{for } 1 \leq i \leq I \\ \frac{u_1^n - u_0^n}{\Delta x} = 0 \\ \frac{u_{I+1}^n - u_I^n}{\Delta x} = 0 \end{cases}$$

- Note that we iterate backward (hence the "Kolmogov Backward" name)

Matrix Version

- This finite scheme can be rewritten as

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^{n}}{\Delta t} = \mathbb{A}' \mathbf{u}^{n}$$

where $\mathbf{u}^n$ is the vector $(u_1^n, u_2^n, \ldots, u_I^n)$ and $\mathbb{A}$ is the following sparse matrix:

$$\mathbb{A} = \begin{bmatrix} -\frac{\mu_1}{\Delta x} - \frac{1}{2}\frac{\sigma_1^2}{(\Delta x)^2} & \frac{1}{2}\frac{\sigma_2^2}{(\Delta x)^2} & & & \\ \frac{\mu_1}{\Delta x} + \frac{1}{2}\frac{\sigma_1^2}{(\Delta x)^2} & -\frac{\mu_2}{\Delta x} - \frac{\sigma_2^2}{(\Delta x)^2} & \ddots & & \\ & \frac{\mu_2}{\Delta x} + \frac{1}{2}\frac{\sigma_2^2}{(\Delta x)^2} & \ddots & -\frac{\mu_{I-1}}{\Delta x} + \frac{1}{2}\frac{\sigma_{I-1}^2}{(\Delta x)^2} & \\ & & \ddots & \frac{\mu_{I-1}}{\Delta x} - \frac{\sigma_{I-1}^2}{(\Delta x)^2} & -\frac{\mu_I}{\Delta x} + \frac{1}{2}\frac{\sigma_I^2}{(\Delta x)^2} \\ & & & \frac{1}{2}\frac{\sigma_{I-1}^2}{(\Delta x)^2} & \frac{\mu_I}{\Delta x} - \frac{1}{2}\frac{\sigma_I^2}{(\Delta x)^2} \end{bmatrix}$$

I have assumed $\mu_1, \mu_2 \geq 0$ and $\mu_{I-1}, \mu_I \leq 0$.
Note that $\mathbb{A}$ is a (i) tridiagonal matrix (ii) its columns sum to zero (iii) with positive off-diagonal elements.

Matrix Version (Implicit Scheme)

- It is often more robust to use the implicit-scheme

$$\frac{\mathrm{u}^{n+1} - \mathrm{u}^{n}}{\Delta t} = \mathbb{A}' \mathrm{u}^{n+1}$$
$$\Rightarrow \mathrm{u}^{n+1} = (I - \mathbb{A}' \Delta t)^{-1} \mathrm{u}^{n}$$

- It is a bit slower since it requires to solve a linear system, rather than simply multiplying by a matrix, but it is much more robust

Kolmogorov Forward

Theorem (Kolmogorov Forward)
Take a diffusion process

$$dx_t = \mu(x_t)dt + \sigma(x_t)dZ_t$$

Denote $g_t(x)$ the density of x_t at time t. We have:

$$\frac{dg}{dt} = -\partial_x(\mu(x)g(x)) + \frac{1}{2}\partial_{xx}(\sigma^2(x)g(x))$$

Heuristic Proof

For any function f, $E[f(x_{t+dt})] - E[f(x_t)]$ can be written in two ways

$$\int_{-\infty}^{+\infty} f(x) dg_t(x) dx = \int_{-\infty}^{+\infty} df(x) g_t(x) dx$$

Ito's lemma on the RHS gives

$$\int_{-\infty}^{+\infty} f(x) dg_t(x) dx = \int_{-\infty}^{+\infty} (\mu(x)\partial_x f(x) + \frac{1}{2}\sigma(x)^2 \partial_{xx} f(x)) g_t(x) dx$$

Assume that f decays fast enough as $|x| \to +\infty$ and use integration by parts to obtain

$$\int_{-\infty}^{+\infty} f(x) dg_t(x) dx = \int_{-\infty}^{+\infty} f(x)(-\partial_x(\mu(x) g_t(x)) + \frac{1}{2}\partial_{xx}(\sigma(x)^2 g_t)) dt dx$$

This equality must hold for all f satisfying the conditions above. Therefore, we obtain

$$\frac{dg_t}{dt}(x) = -\partial_x(\mu(x) g_t(x)) + \frac{1}{2}\partial_x^2(\sigma^2(x) g_t)$$

Theorem (Kolmogorov Forward on a Bounded Space)

Take a Markov diffusion process

$$dx_t = \mu(x_t)dt + \sigma(x_t)dZ_t$$

with reflecting boundary at $\underline{x}, \overline{x}$.

Denote $g_t(x)$ *the density of* x_t *at time* t. *We have:*

$$\frac{dg}{dt} = -\partial_x(\mu(x)g(x)) + \frac{1}{2}\partial_{xx}(\sigma^2(x)g(x))$$
$$0 = -\mu(\underline{x})g(\underline{x}, t) + \frac{1}{2}\partial_x(\sigma^2(\underline{x})g(\underline{x}, t))$$
$$0 = -\mu(\overline{x})g(\overline{x}, t) + \frac{1}{2}\partial_x(\sigma^2(\overline{x})g(\overline{x}, t))$$

Numerical Method

- How to solve for g
  1. Define a state space $\mathbf{x} = (x_1, x_2, \ldots, x_I)$ with a step Δx. Define a time step Δt
  2. Define $g_i^n = g_{n\Delta t}(x_i)$. One can solve the PDE using "finite-difference" method with upwinding.

$$\begin{cases} \frac{g_i^{n+1} - g_i^n}{\Delta t} = -\frac{\mu_i^+ g_i^n - \mu_{i-1}^+ g_{i-1}^n}{\Delta x} - \frac{\mu_{i+1}^- g_{i+1}^n - \mu_i^- g_i^n}{\Delta x} + \frac{1}{2} \frac{\sigma_{i+1}^2 g_{i+1}^n + \sigma_{i-1}^2 g_{i-1}^n - 2\sigma_i^2 g_i^n}{(\Delta x)^2} & \text{for } 1 \leq i \leq I \\ 0 = -\mu_0 g_0^n + \frac{1}{2} \frac{\sigma_1^2 g_1^n - \sigma_0^2 g_0^n}{\Delta x} \\ 0 = -\mu_{n+1} g_{I+1}^n + \frac{1}{2} \frac{\sigma_{I+1}^2 g_{I+1}^n - \sigma_I^2 g_I^n}{\Delta x} \end{cases}$$

- In matrix form, this finite-scheme can be written:

$$\frac{\mathbf{g}^{n+1} - \mathbf{g}^n}{\Delta t} = \mathbb{A} \mathbf{g}^n$$

  where $\mathbf{g}^n j$ is the vector $(g_1^n, g_2^n, \ldots, g_n^n)$ and $\mathbb{A}$ is the matrix defined above.
  Note the relationship between the Kolmogogorov Forward and the Kolmogorov Backward operator. In the set of functions, one is obtained from the other through an integration by part. In the set of vectors (functions on a grid), one is obtained from the other through a matrix transpose.
- Similarly to what we did for Kolmogorov Forward, using an implicit finit-scheme is more robust:

$$\frac{\mathbf{g}^{n+1} - \mathbf{g}^n}{\Delta t} = \mathbb{A} \mathbf{g}^{n+1}$$

Solving for Stationary Distribution

Theorem (Kolmogorov Forward)

The stationary density (if it exists) is the solution to

$$0 = -\partial_x(\mu(x)g(x)) + \frac{1}{2}\partial_{xx}(\sigma^2(x)g(x))$$

with

$$g(x) \geq 0$$
$$\int g(x)dx = 1$$

Numerical Solution

- Numerically, we can solve for the stationary distribution by solving

$$\mathbb{A}\mathbf{g} = 0$$
$$\mathbf{g} \geq 0$$

- Note that the sum of columns of $\mathbb{A}$ is zero, and therefore $\mathbf{e} = (1, 1, \ldots, 1)$ is a right eigenvector of $\mathbb{A}'$ with eigenvalue 0. Therefore there is a non-null right eigenvector of $\mathbb{A}$ with eigenvalue 0, i.e. there is a non-null $\mathbf{g}$ that solves $\mathbb{A}\mathbf{g} = 0$.
- One can show that there is one and only one eigenvector $\mathbf{g}$ that is nonnegative elementwise, i.e. $\mathbf{g} \geq 0$ (this is due to Perron-Frobenius theorem).

- Consider the process

$$dx_t = -\kappa(x_t - \overline{x})dt + \sqrt{x_t}\sigma dZ_t$$

with $\overline{x} = 0.1$, $\kappa = 0.9$ and $\sigma = 0.01$.

- Simulate the process at monthly frequency. Start with $x_0 = 0.1$, and then iterate using the discrete time approximation

$$x_{t+\Delta t} - x_t = -\kappa(x_t - \overline{x}) \times \Delta t + \sqrt{x_t}\sigma \times \sqrt{\Delta t} \times \epsilon$$

with $\Delta t = 1/12$ and $\epsilon \sim N(0, 1)$)

- Using Kolmogorov Forward, plot the stationary density of x_t. On the same graph, plot the empirical stationary distribution obtained using your simulation.