OLG Models

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A Perpetual Youth Model

Endowment

- Denote Y_t the aggregate endowment

$$\frac{dY_t}{Y_t} = \mu dt + \sigma dW_t$$

  - A proportion ω is given as labor
  - A proportion $1 - \omega$ is given as dividends

Death and Birth

- Agents die with harzard rate δ ("perpertual youth" model)
- The mass at time t of people born at time s is $\delta e^{-\delta(t-s)}$
  The total mass of households is $\int_{-\infty}^{t} \delta e^{-\delta(t-s)} ds = 1$
- When agents die, their wealth is redistributed to every existing agent, in proportion to their wealth

Labor Income

- Agent born at s receives at t labor income $L_{st}dt$ (if still living)

$$L_{st} = \omega B e^{-\delta_l (t-s)} Y_t$$

where B chosen so that total labor income is ωY_t

$$\int_{s=-\infty}^{t} \delta e^{-\delta(t-s)} L_{st} ds = \int_{s=-\infty}^{t} \delta e^{-\delta(t-s)} \omega B e^{-\delta_l (t-s)} Y_t ds$$
$$= \frac{\delta}{\delta + \delta_t} \omega B Y_t$$

i.e. $B = (\delta + \delta_l)/\delta$

Financial Markets

- There is interest rate r and market price of risk κ
- Denote P^l_{st} the value at t of a claim of future labor income for an agent born at time t

$$E[\frac{L_{st}}{P^l_{st}}dt + \frac{dP^l_{st}}{P^l_{st}} - \delta dt] = rdt + \kappa\sigma_{P^l}dt$$

δdt accounts for the probability of dying

- Guess there is a constant V^l such that $P^l_{st} = V^l e^{-\delta_l(t-s)}Y_t$. We have:

$$\frac{\omega\frac{\delta+\delta_l}{\delta}}{V^l}dt + E[\frac{e^{-\delta_l(t-s)}Y_t}{e^{-\delta_l(t-s)}Y_t}] - \delta dt = rdt + \kappa\sigma dt$$

$$\Rightarrow \frac{\omega\frac{\delta+\delta_l}{\delta}}{V^l} + \mu - \delta_l - \delta = r + \kappa\sigma$$

- Denote N_{st} the networth of an agent born at time s at time t:

$$N_{ss} = V_l Y_t$$
$$\frac{dN_{st}}{N_{st}} = (r + \delta + \kappa\sigma_N)dt + \sigma_N dZ_t$$

Agent Optimization

- Agents have Epstein Zin preferences and die with hazard rate δ
- They choose a consumption ratio $\hat{c}$ and an exposure to aggregate risk σ_W to maximize

$$0 = f(C, U) + E[dU]$$

$$f(C, U) = \frac{\rho + \delta}{1 - \frac{1}{\psi}} \left[\frac{C^{1-\frac{1}{\psi}}}{[(1-\gamma)U]^{\frac{\gamma - \frac{1}{\psi}}{1-\gamma}}} - (1-\gamma)U \right]$$

- Because agents have homogeneous preferences, they all choose the same consumption-to-weath ratio (denoted $1/V^e$) and exposure to aggregate risk. Euler equations

$$\frac{dC_e}{C_e} = \mu_e dt + \sigma_e dZ_t$$

$$\sigma_{C_e} = \frac{\kappa}{\gamma}$$

$$\mu_{C_e} = \psi(r - \rho) + \frac{1+\psi}{2\gamma}\kappa^2$$

$$\frac{1}{V^e} + \mu_{C_e} - \delta = r + \kappa\sigma_{C_e}$$

Solving the Model

- Denote C_{ts} the consumption at t of agent born at time s:

$$Y_t = \int_{-\infty}^{t} \delta e^{-\delta(t-s)} C_{ts}$$

- Using Ito's lemma

$$dY_t = \underbrace{\int_{-\infty}^{t} \delta e^{-\delta(t-s)} dC_{ts}}_{\text{Consumption Change Existing}} - \underbrace{\delta dt Y_t}_{\text{Consumption due to Dead}} + \underbrace{\delta dt C_{tt}}_{\text{Consumption Arriving}}$$

$$\Rightarrow \frac{dY_t}{Y_t} = \frac{dC_e}{C_e} - \delta dt + \delta dt \frac{C_{tt}}{Y_t}$$

$$= \frac{dC_e}{C_e} + \delta dt(\frac{V^l}{V^e} - 1)$$

- We obtain

$$\mu = \mu_e + \delta(\frac{V^l}{V^e} - 1)$$

$$\sigma = \sigma_e$$

Solving the Model

- The system of equation is

$$r = \rho + \frac{1}{\psi}(\mu - \delta(\frac{V^l}{V^e} - 1)) - (1 + \frac{1}{\psi})\gamma\frac{\sigma^2}{2} \tag{1}$$

$$\kappa = \gamma\sigma \tag{2}$$

$$V^l = \omega\frac{\delta + \delta_l}{\delta}\frac{1}{r + \kappa\sigma - (\mu - \delta - \delta_l)} \tag{3}$$

$$V^e = \frac{1}{r + \kappa\sigma - (\psi(r - \rho) + \frac{1+\psi}{2}\gamma\sigma^2 - \delta)} \tag{4}$$

Consequences

Split Between Existing vs Arriving Matters for Prices

- In this model the interest rate does not only depend on the growth, but also on how consumption is split
- Higher labor share $\Rightarrow$ Wealth of arriving households increases relative to existing households $\Rightarrow$ Consumption growth of existing households diminishes $\Rightarrow$ interest rate decreases

Amplifications

- Equation (1) says the interest rate r decreases in V^l/V^e
- Dependensing on the exact parameter, Equation (3) and Equation (4) say that V_l/V^e may decrease in interest rate

Wealth Distribution

- If $\mu_e \geq \mu$, the distribution of relative consumption in this economy has a Pareto tail

$$\zeta = \frac{\delta}{\mu_e - \mu}$$

- Remember

$$\mu = \mu_e + \delta(\frac{V^l}{V^e} - 1)$$

Therefore

$$\zeta = \frac{1}{1 - \frac{V^l}{V^e}}$$

Homework

- Solve the model i.e. solve for V_l, V_e, r, κ that satisfy Equation (1), Equation (2), Equation (3), Equation (4), using the following parameters: $\mu = 3\%, \sigma = 4\%, \delta_l = 1\%, \omega = 0.7, \rho = 0, \psi = 1.0, \gamma = 2.0, \delta = 2\%,$.
- What is the pareto tail of the wealth distribution in this economy?
- Now solve for the model with $\omega = 0.6$. What happens to the interest rate? What happens to the Pareto tail? Explain the intuition behind the results.