Heterogeneous Agent Models

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Basic Model

Basic Model

- Aggregate Endowment follows geometric Brownian motion

$$\frac{dY_t}{Y_t} = \mu dt + \sigma dW_t$$

- Assume two types of agents $i \in \{A, B\}$ with Epstin-Zin utility $\gamma_A < \gamma_B$.
- Denote C_i consumption of agent of type $i \in \{A, B\}$. Denote μ_{C_i}, σ_{C_i} the geometric drift and volatility of consumption of agent of type i, i.e.

$$\frac{dC_i}{C_i} = \mu_{C_i} dt + \sigma_{C_i} dW_t$$

- Euler equation for agent $i \in \{A, B\}$ gives

$$\sigma_{C_i} = \frac{\kappa}{\gamma_i} \tag{1}$$

- Total consumption is

$$Y = C_A + C_B$$

  Ito's lemma gives:

$$\frac{dY}{Y} = x\frac{dC_A}{C_A} + (1 - x)\frac{dC_B}{C_B}$$

  where $x = C_A/C$ denotes the share of consumption from agent A.
  In particular,

$$\sigma = x\sigma_{C_A} + (1 - x)\sigma_{C_B} \tag{2}$$

Basic Model

- By plugging FOC Equation (1) into market clearing Equation (2), we obtain

$$\kappa = \frac{\sigma}{\frac{x}{\gamma_A} + \frac{1-x}{\gamma_B}} \tag{3}$$

- Denote μ_X, σ_X the arithmetic drift and volatility of x, i.e.

$$dx = \mu_x dt + \sigma_x dW_t$$

Applying Ito's lemma on $x = C_A/C$, we obtain

$$\begin{aligned}\sigma_X &= x(\sigma_A - \sigma) \\ &= x(1-x)(\sigma_A - \sigma_B)\end{aligned}$$

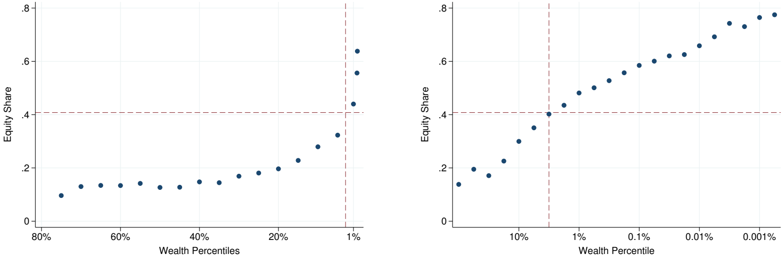
- Applying Ito's lemma on Equation (11), we can show that $\kappa < 0$ is countercyclical:

$$\sigma_\kappa = \frac{\partial \kappa}{\partial x} \times \sigma_X < 0$$

Data

Fact 1: Top Households Invest Twice as Much in Equity as the Representative Household

Figure 1: Equity Share = $\frac{\text{Public Equity + Private Equity}}{\text{Wealth}}$



Data source: Survey of Consumer Finance (SCF), a cross sectional survey of US households from 1989 to 2013. Horizontal line represents the average equity share. Vertical line splits the sample in two of same aggregate wealth.

Fact 2: When Stock Returns are High, Wealth Inequality Increases $\sigma_A - \sigma > 0$

$$\log \frac{\text{Top Wealth}_{t+1}}{\text{Top Wealth}_{t-1}} - r_t^f = \alpha + \beta \times (r_t^M - r_t^f) + \gamma \times r_t^f + \epsilon_t$$

	Groups of Households Defined by Wealth Percentiles				
	Flow of Funds	Estate Tax Returns			Forbes
	All Households	$1 - 0.1\%$	$0.1 - 0.01\%$	Top 0.01%	Top 100
	(1)	(2)	(3)	(4)	(5)
Excess Stock Returns	0.44***	0.52***	0.66***	0.75***	0.71***
	(0.13)	(0.18)	(0.17)	(0.23)	(0.18)
R^2	0.49	0.45	0.58	0.40	0.34
Period	1917-1999	1917-1999	1917-1999	1917-1999	1983-2014
N	54	54	54	54	32

Estimation via OLS. Standard errors in parentheses and estimated using Newey-West with 3 lags. *,** ,*** indicate significance at the 0.1, 0.05, 0.01 levels.

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$$\log \frac{\text{Top Wealth Share}_{t+1}}{\text{Top Wealth Share}_{t-1}} - r_t^f = \alpha + \beta \times (r_t^M - r_t^f) + \gamma \times r_t^f + \epsilon_t$$

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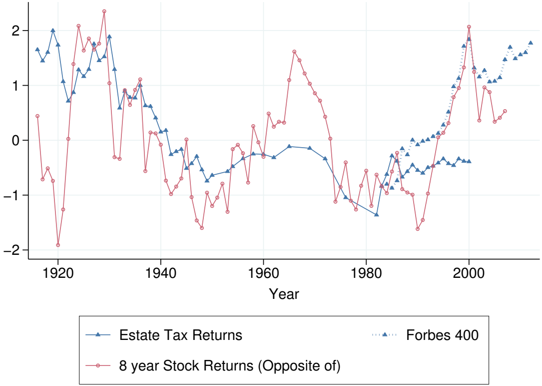
	All Households	$1 - 0.1\%$	$0.1 - 0.01\%$	Top 0.01%	Top 100
	(1)	(2)	(3)	(4)	(5)
Excess Stock Returns		0.09*	0.22***	0.31**	0.27*
		(0.05)	(0.07)	(0.14)	(0.19)
R^2		0.20	0.33	0.14	0.18
Period		1917-1999	1917-1999	1917-1999	1983-2013
N		54	54	54	31

Estimation via OLS. Standard errors in parentheses and estimated using Newey-West with 3 lags. *,** ,*** indicate significance at the 0.1, 0.05, 0.01 levels.

Fact 3: High Wealth Inequality Predicts Lower Future Excess Returns $\partial\kappa/\partial x < 0$

Figure 2: The Wealth Share of the Top 0.01% and Average Excess Returns



Notes. The figure plots the wealth share of the top 0.01% (log) from Estate Tax Returns and from Forbes 400 and the (opposite of) 8-year sum of future excess returns. All series are normalized to have a standard deviation of one.

Garleanu Panageas (2016)

Ingredients

Like the baseline model, but different on two dimensions:

- To make the model stationary, death rate δ. Agents that are born inherit the wealth of deceased households. Denote ν_A the population rate of agent of type A is ν_A
- Agents have Epstein Zin preferences. Agents can differ with respect to RRA $\gamma_A < \gamma_B$ and EIS $\psi_A \neq \psi_B$

Euler Equation

- Agents choose a consumption ratio $\hat{c}$ and an exposure to aggregate risk σ_W to maximize

$$0 = f_i(C, U) + E[dU]$$

$$f(C, U) = \frac{1}{1-\frac{1}{\psi_i}}\left[\frac{C^{1-\frac{1}{\psi_i}}}{[(1-\gamma_i)U]^{\frac{\gamma-\frac{1}{\psi_i}}{1-\gamma_i}}} - (\rho+\delta)(1-\gamma_i)U\right]$$

  where the law of motion of net worth is

$$\frac{dN_{it}}{N_{it}} = ((r+\delta+\kappa\sigma_{Nit})N_{it} - C)dt + \sigma_{Nit}N_{it}dt$$

- Because agents have homogeneous preferences, they all choose the same consumption-to-weath ratio (denoted $1/V_i$) and exposure to aggregate risk. Euler equations

$$\frac{dC_i}{C_i} = \mu_{C_i}dt + \sigma_{C_i}dZ_t$$

  with

$$\sigma_{C_i} = \frac{\kappa}{\gamma_i} + \frac{1-\gamma_i\psi_i}{\gamma_i(\psi_i-1)}\sigma_{V_i} \quad (4)$$

$$\mu_{C_i} = \psi_i(r-(\rho+\delta)) + \frac{1+\psi_i}{2\gamma_i}\kappa^2 + \frac{1-\gamma_i\psi_i}{\gamma_i(\psi_i-1)}\kappa\sigma_{V_i} - \frac{1-\gamma_i\psi_i}{2(\psi_i-1)\gamma_i}\sigma_{V_i}^2 \quad (5)$$

- Denote ν_A the share of population of type A. Denote C_{ts} the consumption at t of agent born at time s:

$$Y_t = \nu_A \int_{-\infty}^{t} \delta e^{-\delta(t-s)} C_{Ats} + (1-\nu_A) \int_{-\infty}^{t} \delta e^{-\delta(t-s)} C_{Bts}$$

- Using Ito's lemma

$$dY_t = \underbrace{\nu_A \int_{-\infty}^{t} \delta e^{-\delta(t-s)} dC_{Ats} + (1-\nu_A) \int_{-\infty}^{t} \delta e^{-\delta(t-s)} dC_{Bts}}_{\text{Consumption Change due to Existing}}$$

$$- \underbrace{\delta dt Y_t}_{\text{Consumption due to Dead}} + \underbrace{\delta dt(\nu_A C_{Att} + (1-\nu_A) C_{Btt})}_{\text{Consumption due to Newborn}}$$

- Denote x the share of consumption by agents in group A

$$x_t = \frac{\nu_A \int_{-\infty}^{t} \delta e^{-\delta(t-s)} C_{Ats}}{Y_t}$$

We can express the growth in endowment in term of individual consumption growth:

$$\frac{dY_t}{Y_t} = x\frac{dC_A}{C_A} + (1-x)\frac{dC_B}{C_B} + \delta dt\left(\nu_A \frac{C_{Att}}{Y_t} + (1-\nu_A)\frac{C_{Btt}}{Y_t} - 1\right)$$

- Denote the wealth to consumption ratio of the economy

$$V = xV_A + (1-x)V_B$$

We have

$$\frac{dY_t}{Y_t} = x\frac{dC_A}{C_A} + (1-x)\frac{dC_B}{C_B} + \delta dt\left(\nu_A \frac{V}{V_A} + (1-\nu_A)\frac{V}{V_B} - 1\right)$$

- We obtain

$$\sigma = x\sigma_{C_A} + (1-x)\sigma_{C_B} \tag{6}$$

$$\mu = x\mu_{C_A} + (1-x)\mu_{C_B} + \delta\left(\nu_A \frac{V}{V_A} + (1-\nu_A)\frac{V}{V_B} - 1\right) \tag{7}$$

- Plugging expressions for $\sigma_{C_A}, \sigma_{C_B}, \mu_{C_A}, \mu_{C_B}$ using Euler equations, we obtain

$$\kappa = \frac{1}{\sum \frac{x_i}{\gamma_i}}(\sigma - \sum x_i \frac{1-\gamma_i\psi_i}{\gamma_i(\psi_i - 1)}\sigma_{V_i}) \tag{8}$$

$$r = \rho + \delta + \frac{1}{\sum x_i\psi_i}(\mu - \delta\left(\nu_A \frac{V}{V_A} + (1-\nu_A)\frac{V}{V_B} - 1\right) \tag{9}$$

$$- \sum x_i(\frac{1+\psi_i}{2\gamma_i}\kappa^2 + \frac{1-\gamma_i\psi_i}{\gamma_i(\psi_i-1)}\kappa\sigma_{V_i} - \frac{1-\gamma_i\psi_i}{2(\psi_i-1)\gamma_i}\sigma_{V_i}^2)) \tag{10}$$

- Finally, Market pricing gives

$$0 = \frac{1}{V_i} + \mu_{C_i} + \mu_{V_i} + \sigma_{C_i}\sigma_{V_i} - r - \kappa(\sigma_{C_i} + \sigma_{V_i}) \tag{11}$$

- Remember x the share of consumption by agents in group A

$$x_t = \frac{\nu_A \int_{-\infty}^{t} \delta e^{-\delta(t-s)} C_{Ats}}{Y_t}$$

- Applying Ito's lemma

$$\frac{dx}{x} = \frac{dC_A}{C_A} + \delta dt(\frac{\nu_A C_{tt}}{xY_t} - 1) - \frac{dY_t}{dY_t} + \sigma^2 - \sigma_{C_A}\sigma dt$$

- Define μ_x and σ_x the arithmetic drift and volatility of x

$$dx = \mu_x dt + \sigma_x dW_t$$

We have

$$\sigma_x = x(\sigma_{C_A} - \sigma) \tag{12}$$

$$\mu_x = x(\mu_{C_A} - \mu + \sigma^2 - \sigma_{C_A}\sigma) + \delta\left(\nu_A \frac{V}{V_A} + (1 - \nu_A)\frac{V}{V_B} - 1\right) \tag{13}$$

1. Given the function V_A, V_B:
2. Inject expression for κ Equation (8) in Equation (4) to express σ_{C_A} in term of σ_{V_A} and σ_{V_B}

$$\sigma_{C_A} = \frac{\sum \frac{x_j}{\gamma_j}}{\gamma_A}(\sigma + \frac{1-x}{\gamma_B}(\frac{1-\gamma_A\psi_A}{\psi_A - 1}\sigma_{V_A} - \frac{1-\gamma_B\psi_B}{\psi_B - 1}\sigma_{V_B}))$$

3. Use Ito's lemma $\sigma_{V_i} = \frac{V_i'}{V_i}\sigma_X$ and Equation (12) to express σ_X in term of V_A', V_A, V_B', V_B

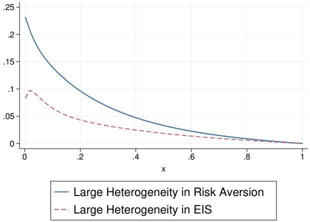
$$\sigma_X = \frac{\frac{x(1-x)\sum \frac{x_i}{\gamma_i}}{\gamma_A\gamma_B}(\gamma_A - \gamma_B)\sigma}{1 - \frac{x(1-x)\sum \frac{x_i}{\gamma_i}}{\gamma_A\gamma_B}(\frac{1-\gamma_A\psi_A}{\psi_A - 1}\frac{V_A'}{V_A} - \frac{1-\gamma_B\psi_B}{\psi_B - 1}\frac{V_B'}{V_B})}$$

4. Given σ_X, we know $\sigma_{V_A}, \sigma_{V_B}, \sigma_{V_I}$ (using Ito's lemma). Therefore we know κ, r, μ_X using Equation (8), Equation (10) and Equation (13)
5. Given μ_X, we know $\mu_{V_A}, \mu_{V_B}, \mu_{V_I}$ (using Ito's lemma), and therefore we can express the system of ODE Equation (11)

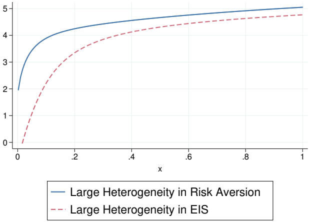
Matching Stock Market Volatility Requires Too Much Heterogeneity

Ito's lemma: $\sigma_{pd} = \sigma_x \times \dfrac{\partial \log pd}{\partial \log x}$

(a) Volatility Wealth Share σ_x



(b) Price-Dividend Ratio $\log pd$



Case 1: Large heterogeneity in risk aversion

Case 2: Large heterogeneity in EIS

In both cases, $\mu_A - \delta \to \mu$ and therefore $\zeta \to 1$