Euler Equation in Discrete Time

Matthieu Gomez

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SDF

The stochastic discount factor

- Denote Θ the set of lotteries (an asset that gives a certain payoff tomorrow).
- Assume Portfolio formation

$$x_1, x_2 \in \Theta \implies ax_1 + bx_2 \in \Theta \; \forall \, a, b, \in \mathbb{R}'$$

- **Def.:** Law of One Price is

$$P(ax_1 + bx_2) = aP(x_1) + bP(x_2)$$

- **Prop:** Law of One Price $\Leftrightarrow$ There exists unique $m \in \Theta$ s.t. $\forall \, X \in \Theta$, $p(x) = \mathbb{E}[mx]$

Proof: $(\Leftarrow)$ is obvious
$(\Rightarrow)$ Let $X = [x_1, \ldots, x_N]'$ a base of Θ (i.e. the set of payoff is $\Theta = \{C'X, C \in R^N\}$).
Then, we just need to find $m = C'X$ such that its prices the basis assets:

$$p(x_1) = E[(C'X)x_1]$$
$$\ldots$$
$$p(x_N) = E[(C'X)x_N]$$

i.e.

$$p(X) = E[(C'X)X]$$

This is obtained by picking C as

$$C = \mathbb{E}[XX']^{-1}p(X)$$

- **Def.:** Absence of arbitrage: $\forall\, x \in \Theta$ s.t. $x \geq 0$ and $x > 0$ with positive probability, $P(x) > 0$.
- **Prop:** Absence of arbitrage $\Leftrightarrow$ There exists unique $m \in \Theta$ s.t. $\forall\, X \in \Theta$, $p(x) = \mathbb{E}[mx]$. Moreover, $m > 0$.

Proof: ($\Leftarrow$) Take payoff $x \in \Theta$ s.t. $x \geq 0$ and $x > 0$ with positive probability, then

$$p(x) = \sum_s \pi(s)m(s)x(s) > 0$$

($\Rightarrow$) Define $\mathcal{M} \equiv \{(-p(x), x) : x \in \Theta\}$. This set only intersects $\mathbb{R}^{s+1}_{+}$ at zero. Thus, from the Separating Hyperplane theorem, $\exists$ a continuous linear function F such that:

$$F(-p, x) = 0 \; \forall \; (-p, x) \in \mathcal{M}$$
$$F(-p, x) > 0 \; \forall \; (-p, x) \in \mathbb{R}^{s+1}_{++}$$

From the Riez representation theorem, $\exists \; m \in \mathbb{R}^{s+1}$ s.t. $F(-p, x) = (1, m) \cdot (-p, x)$ $\forall \; (-p, x)$. This in turn implies that $(1, m) \cdot (-p, x) > 0 \; \forall \; (-p, x) \in \mathbb{R}^{s+1}_{++} \Rightarrow \; m > 0$.

Risk Premia

- For an asset with payoff X_{it+1}, its price P_{it} must be such that:

$$P_{it} = \mathbb{E}_t[M_{t+1}X_{it+1}]$$
$$\Rightarrow 1 = \mathbb{E}_t[M_{t+1}R_{it+1}]$$

- We can rewrite the pricing equation as:

$$1 = \mathbb{E}_t[M_{t+1}]\mathbb{E}_t[R_{it+1}] + cov_t(M_{t+1}, R_{it+1}) \tag{1}$$

- Applyting this to $R_{f,t+1}$, we get

$$R_{f,t+1} = \frac{1}{E_t[M_{t+1}]} \tag{2}$$

- Plugging (2) into (1), we obtain

$$R^f_{t+1} = \mathbb{E}_t[R_{it+1}] + \frac{cov_t(M_{t+1}, R_{it+1})}{\mathbb{E}_t[M_{t+1}]}$$
$$\Rightarrow \mathbb{E}_t R_{it+1} - R^f_{t+1} = -cov_t\left(\frac{M_{t+1}}{\mathbb{E}_t[M_{t+1}]}, R_{it+1}\right)$$

Factor model

- We obtain a one factor model

$$\mathbb{E}_t R_{it+1} - R^f_{t+1} = -\underbrace{\frac{cov_t(M_{t+1}, R_{it+1})}{Var_t(M_{t+1})}}_{\beta_{it}} \underbrace{\frac{Var_t(M_{t+1})}{E_t[M_{t+1}]}}_{\lambda_t}$$

  - β_{it} is the exposure to the systematic risk factor
  - λ_t is the price of risk associted with the factor

Euler Equation

Tying SDF to Consumption

- Assume utility is

$$V_t(W_t) = \max_{C_t,\ldots} \sum_{t}^{+\infty} \beta^t u(C_t), \text{ with } W_{t+1} = R_{t+1}(W_t - C_t)$$

- We can rewrite the problem dynamically

$$V_t(W_t) = \max_{C_t} \{u(C_t) + \beta E_t[V_{t+1}(W_{t+1})]$$

- Deriving the dynamic formulation w.r.t. C_t and W_t

$$\partial C_t : 0 = u'(C_t) - \beta E_t[R_{t+1} V'_{t+1}(W_{t+1})]$$
$$\partial W_t : V'_t(W_t) = \beta E_t[R_{t+1} V'_{t+1}(W_t)]$$

  Therefore

$$u'(C_t) = V'_t(W_t)$$
$$u'(C_t) = \beta E_t[R_{t+1} u'(C_{t+1})]$$

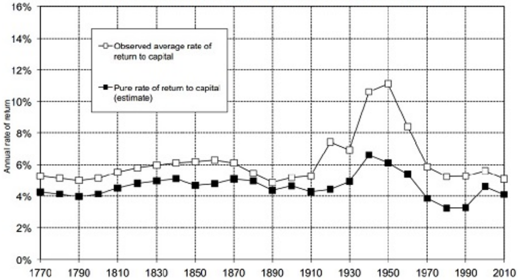
- We obtain the Euler Equation:

$$E[\beta \frac{u'(C_{t+1})}{u'(C_t)} R_{t+1}] = 1$$

  That is, the SDF is

$$M_{t+1} = \beta u'(C_{t+1})/u'(C_t)$$

Figure 6.3. The pure return to capital in Britain, 1770-2010



The pure rate of return to capital is roughly stable around 4%-5% in the long run.
Sources and series: see piketty.pse.ens.fr/capital21c.

- Rate of Return does not display drift $\Longrightarrow$ this means $u'(Ce^g)/u'(C)$ to be constant for all c. At the first order, this means $gu''(c)c/u'(c)$ constant.
- Parenthesis (Gabaix 2008): "In general, asking "what would happen if the firms was 10 times larger?" (or the employee 10 richer), and thinking about which quantities ought not to change (e.g. the interest rate), leads to rather strong constraints on the functional forms in economics"
- Assume CRRA utility

$$u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}$$

- Then

$$M_{t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma}$$

Euler equation is

$$1 = \mathbb{E}_t \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{t+1} \right]$$

Assuming Log Linear Returns and Consumption

- In logs, denote

$$r_{t+1} = \log R_{t+1}$$
$$c_t = \log C_t$$

Euler Equation can be written

$$1 = \mathbb{E}_t[e^{\log\beta - \gamma\Delta c_{t+1} + r_{t+1}}] \tag{3}$$

- Assume consumption and returns are jointly lognormal, i.e.

$$\begin{pmatrix} r_{t+1} \\ \Delta c_{t+1} \end{pmatrix} \sim N\left(\begin{pmatrix} E_t[r_{t+1}] \\ E_t[\Delta c_{t+1}] \end{pmatrix}, \begin{pmatrix} \sigma_r^2 & \sigma_{rc} \\ \sigma_{rc} & \sigma_c^2 \end{pmatrix}\right)$$

- Remember that for a normal variable $x \sim N(E[x], \sigma_x^2)$, we have

$$E[e^x] = e^{E[x] + \frac{1}{2}\sigma_x^2}$$

Euler equation can be rewritten

$$1 = e^{\log\beta - \gamma\mathbb{E}_t\Delta c_{t+1} + \mathbb{E}_t r_{t+1} + \frac{1}{2}(\gamma^2\sigma_c^2 + \sigma_r^2 - 2\gamma\sigma_{rc})}$$

$$\Rightarrow \; 0 = \mathbb{E}_t r_{t+1} + \log\beta - \gamma\mathbb{E}_t\Delta c_{t+1} + \frac{\gamma^2}{2}\sigma_c^2 + \frac{\sigma_r^2}{2} - \gamma\sigma_{rc} \tag{4}$$

- We obtain an expression for the risk free rate:

$$r^f_{t+1} = \underbrace{-\log\beta}_{\text{rate of time pref.}} + \underbrace{\gamma\mathbb{E}_t\Delta c_{t+1}}_{\text{IS term}} - \underbrace{\frac{\gamma^2}{2}\sigma_c^2}_{\text{prec. savings}} \tag{5}$$

- Plugging (5) into (4), we obtain an expression for the equity premium:

$$\underbrace{\mathbb{E}_t r_{t+1} + \frac{\sigma_r^2}{2}}_{\log E_t[R_{t+1}]} - \underbrace{r^f_{t+1}}_{\log R_{ft+1}} = \gamma\sigma_{rc} \tag{6}$$

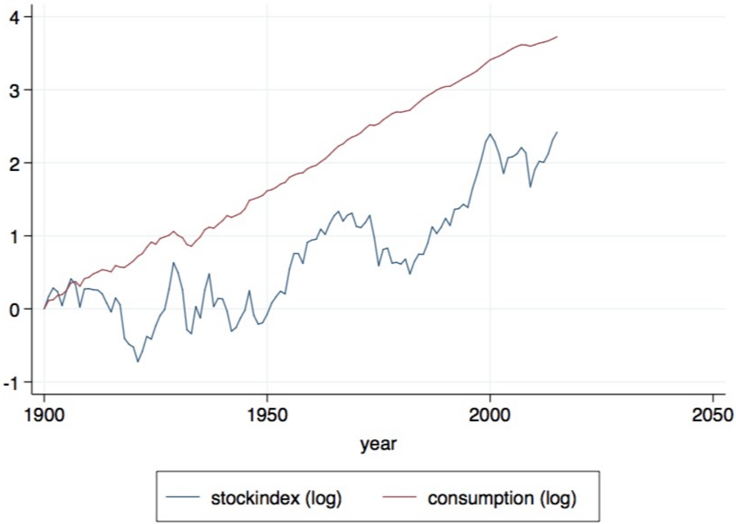
Testing Euler Equation

Table 1
International stock and bill returns

Country	Sample period	$\overline{r_e}$	$\sigma(r_e)$	$\rho(r_e)$	$\overline{r_f}$	$\sigma(r_f)$	$\rho(r_f)$
USA	1947.2–1998.4	8.085	15.645	0.083	0.896	1.748	0.508
AUL	1970.1–1999.1	3.540	22.699	0.005	2.054	2.528	0.645
CAN	1970.1–1999.2	5.431	17.279	0.072	2.713	1.855	0.667
FR	1973.2–1998.4	9.023	23.425	0.048	2.715	1.837	0.710
GER	1978.4–1997.4	9.838	20.097	0.090	3.219	1.152	0.348
ITA	1971.2–1998.2	3.168	27.039	0.079	2.371	2.847	0.691
JAP	1970.2–1999.1	4.715	21.909	0.021	1.388	2.298	0.480
NTH	1977.2–1998.4	14.070	17.228	−0.030	3.377	1.591	−0.085
SWD	1970.1–1999.3	10.648	23.839	0.022	1.995	2.835	0.260
SWT	1982.2–1999.1	13.744	21.828	−0.128	1.393	1.498	0.243
UK	1970.1–1999.2	8.155	21.190	0.084	1.301	2.957	0.478
USA	1970.1–1998.4	6.929	17.556	0.051	1.494	1.687	0.571
SWD	1920–1998	7.084	18.641	0.096	2.209	5.800	0.710
UK	1919–1998	7.713	22.170	−0.023	1.255	5.319	0.589
USA	1891–1998	7.169	18.599	0.047	2.020	8.811	0.338

- Average real returns on stocks are high and have high annualized volatility.
- Average real returns on short-term bills low and have low annualized volatility.

Puzzles


4
3
2
1
0
-1
1900
1950
2000
2050
year
stockindex (log)
consumption (log)

Puzzles

1. Equity Premium is high in the data
    - σ_{rc} is empirically low for stocks, so γ must be very large (i.e., $\hat{\gamma} = 80$) to fit the high average return on stocks.
    - Issues
        - Large γ counterfactual with micro data
        - leads to the risk-free rate puzzle (risk-free rate too big).
2. Equity Premium is time varying in the data
    - σ_{rc} must be time varying

Measuring Consumption

- In real life, consumption happens continuously between t and $t+1$, while returns corresponds to the price at a point in time.

Measuring Consumption

Table II

Alternative Measures of Consumption

Garbage excludes yard trimmings (see Table I). Durables, nondurables, and services are real per capita from NIPA. P–J is the 3-year future consumption expenditure growth as in Parker and Julliard (2005). Q4–Q4 is the fourth-quarter year-over-year consumption expenditure growth as in Jagannathan and Wang (2007). Panel A: R^M is the excess market return. Bootstrapped standard errors are from 10^6 simulations using blocks of size three. The last rows report pairwise correlations. Panel B: Standard errors are Newey–West with three lags.

Panel A: Sample Moments

	Garbage	Durables	Nondurables	Services	Nondur. & Serv.	P–J	Q4–Q4
Mean	1.47	4.62	1.67	2.55	2.21	4.96	2.21
	(0.36)	(0.91)	(0.23)	(0.24)	(0.21)	(0.60)	(0.22)
St. dev.	2.88	5.56	1.45	1.18	1.14	2.99	1.29
	(0.39)	(0.60)	(0.19)	(0.09)	(0.11)	(0.33)	(0.14)
Autocorr.	−14.51	28.90	22.09	51.58	40.01	67.72	32.78
	(11.54)	(11.49)	(12.23)	(11.09)	(10.84)	(6.01)	(11.16)
Corr. R^M	57.94	46.33	47.35	21.89	37.83	13.79	26.42
	(11.25)	(12.00)	(11.58)	(12.11)	(11.64)	(10.53)	(11.47)
Cov. R^M	26.86	41.47	11.04	4.15	6.92	6.64	5.49
	(10.32)	(14.32)	(4.27)	(2.48)	(2.70)	(4.84)	(2.71)
Garbage		42	51	45	53	13	36
Durables			78	57	74	61	65
Nondurables				57	85	54	72
Services					92	42	82
Nondur. & Serv.						53	87
P–J							61

Table I

Sensitivity of Stockholder, Top Stockholder, and Nonstockholder Consumption Growth to Aggregate Consumption Growth Across Horizons

The sensitivity of stockholder, top stockholder, and nonstockholder consumption growth to aggregate consumption growth from NIPA is reported over horizons of $S = 1, 2, 4, 8, 12, 16, 20$, and 24 quarters. The sensitivity of each group's consumption growth to aggregate consumption growth is computed as the regression coefficient from regressing a group's discounted consumption growth over horizon S on aggregate discounted consumption growth over the same horizon. Standard errors (in parentheses) on the regression sensitivity measure are computed using a Newey–West estimator that allows for autocorrelation of up to $S \times 3 - 1$ month lags. Group consumption growth rates are calculated using data from the Consumer Expenditure Survey over the period March 1982 to November 2004.

$S =$	1	2	4	8	12	16	20	24
Stockholder	0.68	0.93	1.21	1.57	2.12	2.68	2.68	2.42
(s.e.)	(0.35)	(0.37)	(0.32)	(0.36)	(0.39)	(0.49)	(0.49)	(0.41)
Top stockholder	0.70	1.01	1.56	2.14	2.88	3.94	3.91	3.48
(s.e.)	(0.90)	(0.77)	(0.62)	(0.49)	(0.53)	(0.67)	(0.73)	(0.63)
Nonstockholder	0.51	0.41	0.59	0.84	0.96	1.01	0.95	0.79
(s.e.)	(0.23)	(0.27)	(0.26)	(0.27)	(0.29)	(0.26)	(0.24)	(0.27)

Figure 2: Long-Run Stockholder Consumption Risk and Asset Returns (2009) JF

They obtain $\gamma \approx 10$ for the wealthiest third of stockholders with the largest holdings of equity

Euler Equation without Consumption Data (Optional)

Euler without Consumption

- Euler Equation for risky asset is

$$\mathbb{E}_t r_{t+1} + \frac{\sigma_r^2}{2} - r_{t+1}^f = \gamma \sigma_{rc} \tag{7}$$

- Since consumption data very noisy, let us try to express the Euler equation in term of wealth data exclusively
- Exposition follows Campbell 1993 “Intertemporal Asset Pricing without Consumption Data”

Using Wealth Data

- Apply Campbell-Schiller to wealth-to-consumption ratio

$$r_{W,t+1} - E_t r_{W,t+1} = c_{t+1} - E_t c_{t+1} + (E_{t+1} - E_t) \sum_{t=1}^{+\infty} \rho_j \Delta c_{t,t+1+j} - (E_{t+1} - E_t) \sum_{t=1}^{+\infty} \rho_j r_{W,t+1+j}$$

- Putting innovation in consumption growth at the LHS,

$$c_{t+1} - E_t c_{t+1} = r_{W,t+1} - E_t r_{W,t+1} - (E_{t+1} - E_t) \sum_{t=1}^{+\infty} \rho_j \Delta c_{t,t+1+j} + (E_{t+1} - E_t) \sum_{t=1}^{+\infty} \rho_j r_{W,t+1+j}$$

- Assume that consumption growth is homoskedastic, i.e. changes in expected returns purely come from changes in expected consumption growth:

$$(E_{t+1} - E_t) r_{Wt+1+j} = \gamma (E_{t+1} - E_t) \Delta c_{t+1+j}$$

  Therefore

$$c_{t+1} - E_t c_{t+1} = r_{W,t+1} - E_t r_{W,t+1} + (1 - \frac{1}{\gamma})(E_{t+1} - E_t) \sum_{t=0}^{+\infty} \rho_j r_{W,t+1+j}$$

- We can therefore write the average excess return of asset i as

$$\mathbb{E}_t r_{it+1} + \frac{\sigma_i^2}{2} - r_{t+1}^f = \gamma \sigma_{iw} + (\gamma - 1) \sigma_{i,(E_{t+1} - E_t) \sum_{t=1}^{+\infty} \rho_j r_{w,t+1+j}} \tag{8}$$

- This version of Euler Equation does not use consumption data

$$\mathbb{E}_t r_{it+1} + \frac{\sigma_i^2}{2} - r_{t+1}^f = \underbrace{\gamma\sigma_{iw}}_{\text{myopic demand}} + \underbrace{(\gamma - 1)\sigma_{i,(E_{t+1}-E_t)\sum_{t=1}^{+\infty} \rho_j r_{w,t+1+j}}}_{\text{hedging demand}} \quad (9)$$

Large changes in expected discount rates can potentially explain

1. high equity premium
2. cross section of returns

- CAPM is defined as

$$\mathbb{E}_t r_{it+1} + \frac{\sigma_i^2}{2} - r_{t+1}^f = \gamma\sigma_{iw} \quad (10)$$

CAPM holds in a word where either

(i) Discount rates are constant
(ii) $\gamma = 1$ (substitution effect = income effect)

In both cases, C/W is constant, and therefore $\sigma_{iw} = \sigma_{ic}$