Distribution

Matthieu Gomez

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Law of Motion of Density

- Take a diffusion process

$$dx_t = \mu(x_t)dt + \sigma(x_t)dZ_t$$

- How does the density of x_t evolves?

Theorem (Kolmogorov Forward)

Denote $g_t(x)$ the density of x_t. We have:

$$\frac{dg}{dt} = -\partial_x(\mu(x)g(x)) + \frac{1}{2}\partial_{xx}(\sigma^2(x)g(x))$$

Heuristic Proof

For any function f, $E[f(x_{t+dt})]$ can be written in two ways

$$\int_{-\infty}^{+\infty} f(x)g_{t+dt}(x)dx = \int_{-\infty}^{+\infty} [(f(x) + df(x))g_t(x)dx$$

Assume that f is a twice differentiable and use Ito's lemma on the RHS to obtain

$$\int_{-\infty}^{+\infty} f(x)dg_t(x)dx = \int_{-\infty}^{+\infty} (\mu(x)\partial_x f(x) + \frac{1}{2}\sigma(x)^2\partial_{xx} f(x))g_t(x)dx$$

Assume that f decays fast enough as $|x| \to +\infty$ and use integration by parts to obtain

$$\int_{-\infty}^{+\infty} f(x)dg_t(x)dx = \int_{-\infty}^{+\infty} f(x)[(-\partial_x(\mu(x)g_t(x)) + \frac{1}{2}\partial_x^2(\sigma(x)^2 g_t))dtdx$$

This equality must hold for all f satisfying the conditions above. Therefore, we obtain

$$\frac{dg_t}{dt}(x) = -\partial_x(\mu(x)g_t(x)) + \frac{1}{2}\partial_x^2(\sigma^2(x)g_t)$$

Find Stationary Density

- Take a diffusion process

$$dx_t = \mu(x_t)dt + \sigma(x_t)dZ_t$$

- **Theorem (Kolmogorov Forward)**
  The stationary density (if it exists) is the solution to

$$0 = -\partial_x(\mu(x)g(x)) + \frac{1}{2}\partial_{xx}(\sigma^2(x)g(x))$$

  with

$$g(x) \geq 0$$
$$\int g(x)dx = 1$$

Solution

- Start from

$$0 = -\partial_x(\mu(x)g(x)) + \frac{1}{2}\partial_{xx}(\sigma^2(x)g(x))$$

Integrating wrt x, we obtain

$$\frac{C_1}{2} = -\mu(x)g(x) + \frac{1}{2}\partial_x(\sigma^2(x)g(x))$$

This is an ODE of degree one.

- The solution has the form:

$$g(x) = m(x)(C_1 S(x) + C_2)$$

with

$$s(x) = \exp(-\int^x \frac{2\mu(\xi)}{\sigma^2(\xi)} d\xi)$$

$$S(x) = \int^x s(\xi) d\xi$$

$$m(z) = \frac{1}{s(x)\sigma^2(x)}$$

and C_1 and C_2 are constant chosen to satisfy

$$g(x) \geq 0$$

$$\int g(x)dx = 1$$

Special Cases

- Take a Ornstein-Uhlenbeck ($\approx$ AR(1) in discrete time)

$$dx = -\kappa(x-\mu)dt + \sigma dZ_t$$

Kolmogorov Forward gives

$$0 = \partial_x(\kappa(x-\mu)g(x)) + \sigma^2 \frac{1}{2}\partial_{xx}(g(x))$$

The solution is a normal distribution

$$g(x) = \sqrt{\frac{\kappa}{\pi\sigma^2}} e^{-\kappa(x-\mu)^2/\sigma^2}$$

- Take a Cox-Ingersoll-Ross process

$$dx = -\kappa(x-\mu)dt + \sigma\sqrt{x}dZ_t$$

Kolmogorov Forward gives

$$0 = \partial_x(\kappa(x-\mu)g(x)) + \sigma^2 \frac{1}{2}\partial_{xx}(xg(x))$$

The solution is a gamma distribution

$$g(x) = \frac{\omega^\nu}{\Gamma(\nu)} x^{\nu-1} e^{-\omega x}$$

with $\omega = 2\kappa/\sigma^2$ and $\nu = 2\kappa\mu/\sigma^2$

Solving Kolmogorov Forward Analytically

Solving ODE

- Alternatively, we can solve with a computer the ODE for $x \in [\underline{x}, \overline{x}]$

$$0 = -\partial_x(\mu(x)g(x)) + \frac{1}{2}\partial_x x\sigma^2(x)g(x)$$

with boundary conditions

$$0 = -\mu(\underline{x})g(\underline{x})) + \frac{1}{2}\partial_x\sigma^2(\underline{x})g(\underline{x})$$
$$0 = -\mu(\overline{x})g(\overline{x})) + \frac{1}{2}\partial_x\sigma^2(\overline{x})g(\overline{x})$$

- Solution method
  1. Define a state space $\mathbf{s} = (s_1, s_2, \ldots, s_n)$.
  2. The problem is to find g positive elementwise such that

$$Ag = 0$$

- To approximate $-\partial_x(\mu(x)g(x))$, use the following matrix

$$-\partial_x(\mu(x)g(x)) = D^1 g$$

where D^1 is the matrix

$$D^1 = \begin{bmatrix} -\frac{\mu_1}{\Delta s} & 0 & 0 & \dots & 0 & 0 & 0 \\ \frac{\mu_1}{\Delta s} & -\frac{\mu_2}{\Delta s} & 0 & \dots & 0 & 0 & 0 \\ 0 & \frac{\mu_2}{\Delta s} & -\frac{\mu_3}{\Delta s} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{\mu_{n-2}}{\Delta s} & -\frac{\mu_{n-1}}{\Delta s} & 0 \\ 0 & 0 & 0 & \dots & 0 & \frac{\mu_{n-1}}{\Delta s} & -\frac{\mu_n}{\Delta s} \\ 0 & 0 & 0 & \dots & 0 & 0 & \frac{\mu_n}{\Delta s} \end{bmatrix}$$

(assuming $\mu_1 \geq 0$ and $\mu_n \leq 0$). Formally

$$-\partial_x(\mu(x)g(x))_i = -\frac{\mu_i^+ g_i - \mu_{i-1}^+ g_{i-1}}{\Delta s} - \frac{g_{i+1}\mu_{i+1}^- - g_i\mu_i^-}{\Delta s}$$

where $\mu_i^+ = max(\mu_i, 0)$ and $\mu_i^- = min(\mu_i, 0)$

Solving ODE

- To approximate $\partial_{xx}(\sigma^2(x)g(x))$, use the following matrix

$$\frac{1}{2}\partial_{xx}(\sigma^2(x)g(x)) = D^2 g$$

where D^2 is the matrix

$$D^2 = \begin{bmatrix} -\frac{\sigma_1^2}{(\Delta s)^2} & \frac{\sigma_2^2}{2(\Delta s)^2} & 0 & \dots & 0 & 0 & 0 \\ \frac{\sigma_1^2}{2(\Delta s)^2} & -2\frac{\sigma_2^2}{2(\Delta s)^2} & \frac{\sigma_3^2}{2(\Delta s)^2} & \dots & 0 & 0 & 0 \\ 0 & \frac{\sigma_2^2}{2(\Delta s)^2} & -2\frac{\sigma_3^2}{2(\Delta s)^2} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -2\frac{\sigma_{n-2}^2}{2(\Delta s)^2} & \frac{\sigma_{n-1}^2}{2(\Delta s)^2} & 0 \\ 0 & 0 & 0 & \dots & \frac{\sigma_{n-2}^2}{2(\Delta s)^2} & -2\frac{\sigma_{n-1}^2}{2(\Delta s)^2} & \frac{\sigma_n^2}{2(\Delta s)^2} \\ 0 & 0 & 0 & \dots & 0 & \frac{\sigma_{n-1}^2}{2(\Delta s)^2} & -\frac{\sigma_n^2}{2(\Delta s)^2} \end{bmatrix}$$

Formally

$$\frac{1}{2}\partial_{xx}(\sigma^2(x)g(x)) = \frac{\sigma_{i+1}^2 g_{i+1} + \sigma_{i-1}^2 g_{i-1} - 2\sigma_i^2 g_i}{2(\Delta s)^2}$$

Solving ODE

- We need to find a vector g such that

$$(D^1 + D^2)g = 0 \tag{1}$$

- How can we be sure that there exists a solution g that is positive everywhere?

Theorem

Suppose a square matrix A is such that

1. *Sum of each column is 0*
2. *All elements off-diagonal are positive or null*

Then there exists a unique $g \geq 0$ such that

$$Ag = 0$$

Proof.

Take a positive real number δ. The matrix $P = \delta I + A$ is such that

1. sum of each column is 1
2. all elements are positive or null (for δ high enough)

Therefore, we know (existence stationary distribution for markov chain in discrete time) there is a unique $g \geq 0$ such that

$$Pg = g$$

In particular, this means there exists a unique $g \geq 0$ such that

$$Ag = 0$$

□

Homework

- Simulate the following process process at monthly frequency

$$dx_t = -\theta(x_t - \overline{x})^3 dt + \sigma dZ_t$$

with $\overline{x} = 0$, $\theta = 0.9$ and $\sigma = 0.2$, starting from $x_0 = 0$
Tip: iterate using the discrete time approximation:

$$x_{t+\Delta t} - x_t = -\theta(x_t - \overline{x})^3 \times \Delta t + \sigma \times \sqrt{\Delta t} \times \epsilon_t$$

with $\Delta t = 1/12$ and $\epsilon \sim N(0,1)$)
- Plot the empirical distribution (as an histogram) of the values of x_t across time
- Compute the stationary density using Kolmogorov Forward theorem (Equation (1). Compare with the empirical stationary distribution obtained using simulation.

Power Law Distribution

Power Law

- Wealth follows power law distribution if wealth density g follows

$$g(x) = Cx^{-\zeta-1} \text{ for } x \geq x_{\min}$$

  ζ is called the power law exponent.
  C is pinpointed by the fact density sums to one, which gives

$$g(x) = \frac{\zeta}{x_{\min}}\left(\frac{x}{x_{\min}}\right)^{-\zeta-1}$$

- Note that the nth moment exists only if $\zeta > n$
- When $\zeta = 1$, it is called Zipf's law

Power Law

- For a power law distribution, we have

$$P(X \geq x) = \int_{x_{\min}}^{+\infty} \frac{\zeta}{x_{\min}} (\frac{x}{x_{\min}})^{-\zeta-1} dx$$
$$= (\frac{x}{x_{\min}})^{-\zeta}$$

- This gives a natural way to test for power law distribution: plot $\log P(X \geq x)$ in term of $\log x$

Special Cases

- Suppose that wealth follows geometric Brownian Motion

$$\frac{dx}{x} = \mu dt + \sigma dZ_t$$

In log, using Ito's lemma,

$$d \ln x = (\mu - \frac{\sigma^2}{2})dt + \sigma dZ_t$$

- Kolmogorov Forward gives that the density of $y = \ln x$ follows

$$\partial_t g = -\partial_y(\mu g(y)) + \sigma^2 \frac{1}{2} \partial_{yy}(g(y))$$

- The solution is a normal distribution (i.e. x follows a log normal distribution)

$$g(y,t) = \frac{1}{\sigma_t \sqrt{2\pi}} e^{-\frac{(y-\mu_t)^2}{2\sigma_t^2}}$$

$$\mu_t = (\mu - \frac{1}{2}\sigma^2)t$$

$$\sigma_t = \sigma \sqrt{t}$$

Special Cases

First way to have stationary distribution: assume that the process is reflected if wealth hits some lower bound x_{min}.

- Kolmogorov Forward is

$$0 = -\partial_x(\mu x g(x)) + \sigma^2 \frac{1}{2} \partial_{xx}(x^2 g(x))$$

- Guess

$$g(x) = Cx^{-\zeta-1} \text{ for } x \geq x_{\text{min}}$$

  We obtain that ζ is such that

$$0 = \zeta\mu + \frac{\zeta(\zeta-1)}{2}\sigma^2$$
$$\Rightarrow \zeta = 1 - \frac{2\mu}{\sigma^2}$$

  Note that one needs $\zeta > 0$, that is, $\mu - \sigma^2/2 < 0$ for the wealth distribution to exist

Special Cases

Second way to have stationary distribution: assume that people die with rate δ and are reinjected at a given wealth

- Kolmogorov Forward is

$$0 = -\partial_x(\mu x g(x)) + \sigma^2 \frac{1}{2} \partial_{xx}(x^2 g(x)) - \delta g(x)$$

- Guess

$$g(x) = C x^{-\zeta - 1}$$

  We obtain

$$0 = \zeta\mu + \frac{\zeta(\zeta - 1)}{2}\sigma^2 - \delta$$

$$\Rightarrow \zeta = \frac{1 - \frac{2\mu}{\sigma^2} + \sqrt{(1 - \frac{2\mu}{\sigma^2})^2 + 8\frac{\delta}{\sigma^2}}}{2}$$

Law of Motion of Top Shares

Diffusion Process

- Assume that wealth of households relative to the average wealth in the economy follows a geometric Brownian Motion:

$$\frac{dx_{it}}{x_{it}} = \mu_t dt + \nu_t dB_{it}$$

  - B_{it} is an idiosyncratic Brownian Motion

- The growth of the share of wealth owned by top percentile p is:

$$\frac{dS_t}{S_t} = \mu_t dt + \frac{g_t(q_t)q_t^2}{2S_t}\nu_t^2 dt \tag{2}$$

  - q_t denotes the wealth threshod at percentile p
  - $g_t(q_t)$ denotes the density of wealth at percentile p

Formal Proof from Kolmogorov Forward

- The top quantile q_t is defined as

$$p = \int_{q_t}^{+\infty} g_t(x)dx$$

During a short period of time dt:

$$0 = \int_{q_t}^{+\infty} dg_t(x)dx - g_t(q_t)dq_t$$

$$\Rightarrow dq_t = \frac{\int_{q_t}^{+\infty} dg_t(x)dx}{g_t(q_t)}$$

- The top wealth share is defined as

$$S_t = \int_{q_t}^{+\infty} xg_t(x)dx$$

During a short period of time dt:

$$\begin{aligned} dS_t &= \int_{q_t}^{+\infty} xdg_t(x)dx - q_t g_t(q_t)dq_t \\ &= \int_{q_t}^{+\infty} xdg_t(x)dx - \int_{q_t}^{+\infty} dg_t(x)dx \\ &= \int_{q_t}^{+\infty} (x - q_t)dg_t(x)dx \end{aligned}$$

Using Kolmoforov Forward and integrating by parts, one obtains Equation (2)

Heuristic Proof

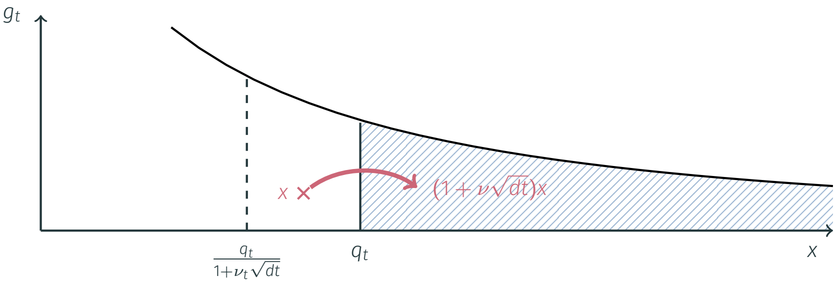
- Alternatively, we can prove the law of motion of top wealth shares heuristically. Take the case $\mu = 0$ to simplify. Let us show that the the entry and exit in top wealth shares sum up to $\frac{g_t(q_t)q_t^2}{2S_t}\nu_t^2 dt$
- The process

$$\frac{dw_t}{w_t} = \nu_t dZ_t$$

can be seen as limit of the process

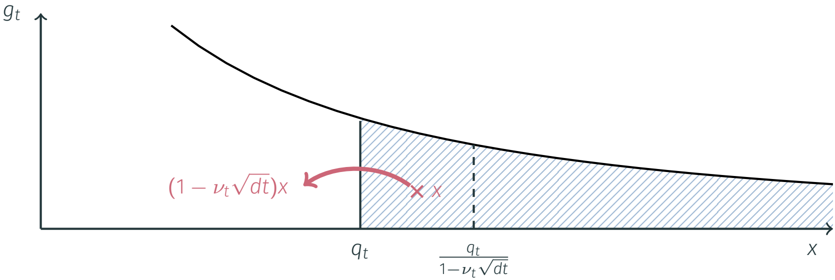
$$\begin{aligned} w_{t+\Delta t} &= (1 + \nu_t\sqrt{\Delta t})w_t \text{ with probability } 1/2 \\ &= (1 - \nu_t\sqrt{\Delta t})w_t \text{ with probability } 1/2 \end{aligned}$$

Heuristic Proof



$$dr_{\text{entry}} \approx \int_{\frac{q_t}{(1+\nu_t\sqrt{dt})}}^{q_t} \frac{(1+\nu_t\sqrt{dt})x - q_t}{2S_t} g_t(x)dx \approx \frac{g_t(q_t)q_t^2}{4S_t}\nu_t^2 dt$$

Heuristic Proof



$$dr_{\text{exit}} \approx \int_{q_t}^{\frac{q_t}{(1-\nu_t\sqrt{dt})}} \frac{q_t-(1-\nu_t\sqrt{dt})x}{2S_t} g_t(x)dx \approx \frac{g_t(q_t)q_t^2}{4S_t}\nu_t^2 dt$$

Pareto Distribution

- Assume that the wealth distribution is Pareto at time t

$$\mathbb{P}_t(x_{it} \geq x) = Cx^{-\zeta}$$

- The growth of top wealth share S_t is:

$$\frac{dS_t}{S_t} = \mu_t dt + \frac{\zeta - 1}{2}\nu_t^2 dt$$

▸ Pareto Tail ▸ Aggregate Shocks ▸ Demographic Forces

Scale Dependence

- Now I allow the geometric drift and volatility of wealth to depend on the wealth level x_{it}:

$$\frac{dx_{it}}{x_{it}} = \mu_t(x_{it})dt + \nu_t(x_{it})dB_{it}$$

- The top wealth share S_t follows the law of motion:

$$\frac{dS_t}{S_t} = E^x[\mu_t(x)|x \geq q_t]dt + \frac{\zeta - 1}{2}\nu_t(q_t)^2 dt$$

Mapping Decomposition to the Data

Accounting Decomposition

- Denote x_{it} the wealth of household i relative to total U.S. wealth
- For a top percentile p, denote top wealth share S_t:

$$S_t = \sum_{i \in \mathcal{T}_t} x_{it}$$

where $\mathcal{T}_t$ denotes the set of households in the top percentile at time t.

Accounting Decomposition

- The growth of S_t between t and $t+1$ can be written as:

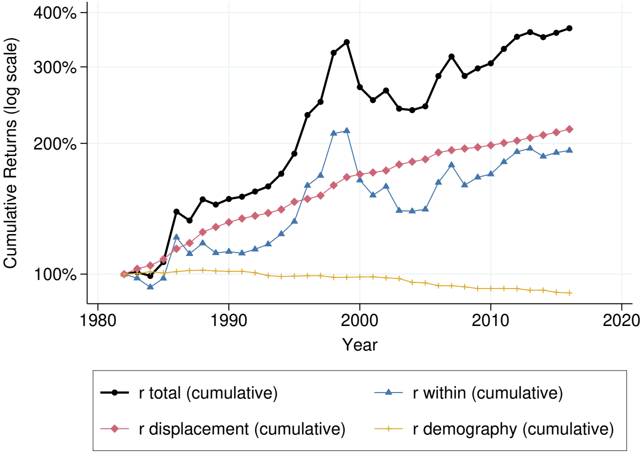
$$\frac{S_{t+1}}{S_t} - 1 = \frac{\sum_{i \in \mathcal{T}_{t+1}} x_{it+1}}{\sum_{i \in \mathcal{T}_t} x_{it}} - 1$$

$$= \frac{\sum_{i \in \mathcal{T}_t} x_{it+1}}{\sum_{i \in \mathcal{T}_t} x_{it}} - 1 + \frac{\sum_{i \in \mathcal{E}} x_{it+1} - \sum_{i \in \mathcal{X}} x_{it+1}}{\sum_{i \in \mathcal{T}_t} x_{it}}$$

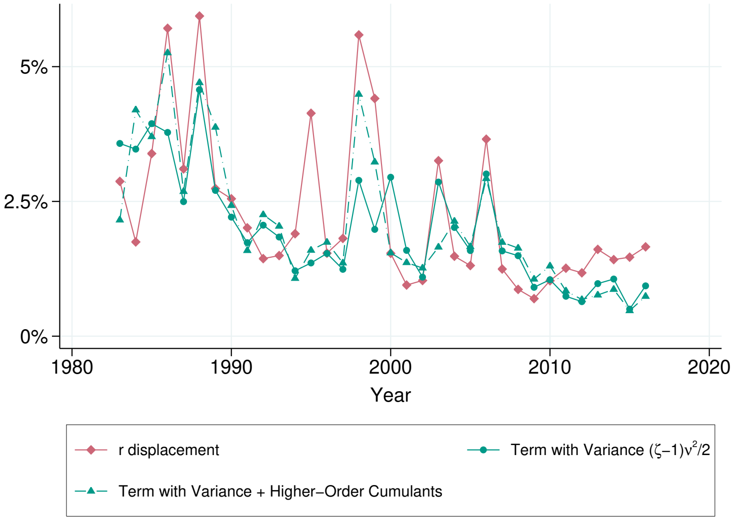
  - $\mathcal{E}$ denotes the set of households that *enter* the top between t and $t+1$
  - $\mathcal{X}$ denotes the set of households that *exit* the top between t and $t+1$

Demographics Measurement Error

Cumulative Decomposition



Notes. The figure plots the cumulative growth of the wealth share of the top 0.0003%, as well as the cumulative return of the within term, the displacement term, and the demography term.



Summary Statistics

Contribution Power Law Exponent ζ vs Idiosyncratic Volatlity ν