Disaster Risk

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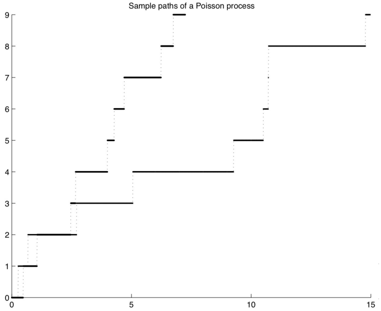
Jump Process

Poisson Process

- A Poisson Process with intensity λ is defined as

$$N_{t+\Delta t} = \begin{cases} N_t + 1 \text{ with probability } \lambda \Delta t \\ N_t \text{ with probability } 1 - \lambda \Delta t \end{cases}$$



Compound Poisson Process

- A *Compound* Poisson Process with intensity λ is defined as

$$Y_{t+\Delta t} = \begin{cases} Y_t + J_t \text{ with probability } \lambda\Delta t \\ Y_t \text{ with probability } 1 - \lambda\Delta t \end{cases}$$

  J_t, the jump size, is itself a random variable
- We write

$$dY_t = J_t dN_t$$

- Note that

$$E[dY_t] = E[J_t]\lambda dt$$

Ito Lemma

- Consider the process x_t

$$dx_t = \mu_t dt + \sigma_t dZ_t + J_t dN_t$$

- Ito's Lemma with jumps. For a twice differentiable function f, we have

$$df(x_t) = \underbrace{f'(x_{t\text{-}})(\mu_t dt + \sigma_t dZ_t) + \frac{1}{2}f''(x_{t\text{-}})\sigma_t^2 dt}_{\text{Diffusive Part}} + \underbrace{(f(x_{t\text{-}} + J_t) - f(x_{t\text{-}}))dN_t}_{\text{Jump Part}}$$

In particular

$$E[df(x_t)] = f'(x_{t\text{-}})\mu_t dt + \frac{1}{2}f''(x_{t\text{-}})\sigma_t^2 dt + E[f(x_{t\text{-}} + J_t) - f(x_{t\text{-}})]\lambda dt$$

Ito Lemma: Example 1

- Consider the process x_t

$$dx_t = \mu_t dt + \sigma_t dZ_t + J_t dN_t$$

- Denote $y_t = e^{x_t}$. Applying Ito's lemma, we have

$$\frac{dy_t}{y_{t\text{-}}} = \mu_t + \sigma_t dZ_t + \frac{\sigma_t^2}{2} dt + (e^{J_t} - 1) dN_t$$

Ito Lemma: Example 2

- Consider the two processes

$$dx_t = \mu_{xt} dt + \sigma_{xt} dZ_t + J_{xt} dN_t$$
$$dy_t = \mu_{yt} dt + \sigma_{yt} dZ_t + J_{yt} dN_t$$

- Then Ito's lemma gives

$$d(x_t y_t) = x_{t\text{-}}(\mu_{yt} dt + \sigma_{yt} dZ_t) + y_{t\text{-}}(\mu_{xt} dt + \sigma_{yt} dZ_t) + \sigma_{xt}\sigma_{yt} dt$$
$$+ ((x_{t\text{-}} + J_{xt})(y_{t\text{-}} + J_{yt}) - x_{t\text{-}} y_{t\text{-}}) dN_t$$

  This can also be written as

$$d(x_t y_t) = x_{t\text{-}} dy_t + y_{t\text{-}} dx_t + \sigma_{xt}\sigma_{yt} dt + J_{xt} J_{yt} dN_t$$

- Denoting $\Delta x_t = J_{xt} dN_t$ and $\Delta y_t = J_{yt} dN_t$, we can also write

$$d(x_t y_t) = x_{t\text{-}} dy_t + y_{t\text{-}} dx_t + \sigma_{xt}\sigma_{yt} dt + \Delta x_t \Delta y_t$$

  This can also be written as

$$\frac{d(x_t y_t)}{x_{t\text{-}} y_{t\text{-}}} = \frac{dy_t}{y_{t\text{-}}} + \frac{dx_t}{x_{t\text{-}}} + \frac{\sigma_{xt}}{x_{t\text{-}}} \frac{\sigma_{yt}}{y_{t\text{-}}} dt + \frac{\Delta x_t}{x_{t\text{-}}} \frac{\Delta y_t}{y_{t\text{-}}}$$

SDF in Continuous Time

- The pricing equation

$$P_t = E_t \int_t^{+\infty} \frac{\Lambda_\tau}{\Lambda_t} D_\tau d\tau$$

Taking the derivative, we obtain

$$d(\Lambda_t P_t) = -\Lambda_t D_t dt \quad (1)$$

- Dividing Equation (1) by $\Lambda_t P_t$ and applying Ito's Lemma, we obtain:

$$E[\frac{d\Lambda_t}{\Lambda_{t\text{-}}}] + E[\frac{dP_t}{P_{t\text{-}}}] + \sigma_P \sigma_\Lambda dt + E[\frac{\Delta P_t}{P_{t\text{-}}} \frac{\Delta \Lambda_t}{\Lambda_{t\text{-}}}] = -\frac{D_t dt}{P_{t\text{-}}}$$

where σ_P and σ_Λ denote the geometric volatility of the processes P and Λ

- Define the instantaneous return as

$$\frac{dR_t}{R_{t\text{-}}} = \frac{D_t dt + dP_t}{P_{t\text{-}}}$$

We can rewrite the market pricing equation:

$$E[\frac{dR_t}{R_{t\text{-}}}] = -E[\frac{d\Lambda_t}{\Lambda_{t\text{-}}}] - \sigma_\Lambda \sigma_R dt - E[\frac{\Delta R_t}{R_{t\text{-}}} \frac{\Delta \Lambda_t}{\Lambda_{t\text{-}}}] \quad (2)$$

Portfolio Problem with Jumps

Consumption process.

Assume the following process for aggregate consumption:

$$\frac{dC_t}{C_{t\text{-}}} = \mu dt + \sigma dZ_t + (e^{J_t} - 1)dN_t$$

where

1. dZ_t is a standard Brownian motion (potentially multi-dimensional)
2. dN_t is a Poisson process with intensity λ_t, which follows a diffusion:

$$d\lambda_t = \mu_\lambda dt + \sigma_\lambda dZ_t$$

Portfolio Problem

- Denote R_t the value of the consumption claim and guess

$$\frac{dR_t}{R_{t\text{-}}} = \mu_R dt + \sigma_R dZ_t + (e^{J_t} - 1)dN_t$$

Note that in particular

$$E[\frac{dR_t}{R_{t\text{-}}}] = \mu_R dt + E[e^{J_t} - 1]\lambda dt$$

- Denote U the utility function. The portfolio problem is

$$0 = \max_{C,\alpha}\{f(C_t, U_t) + E[dU_t]\}$$

$$f(C,U) = \frac{\rho}{1-\frac{1}{\psi}}\left(\frac{C^{1-\frac{1}{\psi}}}{((1-\gamma)U)^{\frac{\gamma-\frac{1}{\psi}}{1-\gamma}}} - (1-\gamma)U\right)$$

$$\frac{dW_t}{W_{t\text{-}}} = \underbrace{(r + \alpha(\mu_R - r) - \frac{C}{W})}_{\mu_W} dt + \alpha\sigma_R dZ_t + \alpha(e^{J_t} - 1)dN_t$$

r is the risk free rate, C_t is consumption, α is share of wealth invested in the risky asset

- Guess there exists a process V_t such that

$$U_t(W) = \frac{W^{1-\gamma}}{1-\gamma} V_t^{\frac{1-\gamma}{\psi-1}} \rho^{\frac{1-\gamma}{1-\frac{1}{\psi}}}$$

- Assume that V_t is a diffusion process with geometric drift μ_V and geometric volatility σ_V, i.e.

$$\frac{dV}{V} = \mu_V dt + \sigma_V dZ_t$$

Note that V is a diffusion. Jumps change the level of consumption, but not the dynammics of consumption going forward $\rightarrow$ we can guess that V is not impacted by jumps, only W is.

FOC Portfolio Problem

- Applying Ito's lemma

$$E[\frac{dW^{1-\gamma}}{W^{1-\gamma}}] = (1-\gamma)\mu_W dt - \frac{\gamma(1-\gamma)}{2}\alpha^2\sigma_R^2 dt + E[(1+\alpha(e^{J_t}-1))^{1-\gamma}-1]\lambda dt$$

- Therefore HJB is

$$0 = \max_{C,\alpha}\Big\{\frac{1}{1-\frac{1}{\psi}}(\frac{C^{1-\frac{1}{\psi}}}{W^{1-\frac{1}{\psi}}V^{\frac{1}{\psi}}}-\rho) + \mu_W + \frac{1}{\psi-1}\mu_V - \frac{\gamma}{2}\alpha^2\sigma_R^2 + \frac{2-\gamma-\psi}{2(\psi-1)^2}\sigma_V^2$$
$$+ \frac{1-\gamma}{\psi-1}\alpha\sigma_R\sigma_V + \frac{1}{1-\gamma}E[(1+\alpha(e^{J_t}-1))^{1-\gamma}-1]\lambda\Big\}$$

- FOCs give

$$\frac{C}{W} = \frac{1}{V}$$
$$0 = \mu_R - r - \gamma\alpha\sigma_R^2 + \frac{1-\gamma}{\psi-1}\alpha\sigma_R\sigma_V + \lambda E[(e^{J_t}-1)(1+\alpha(e^{J_t}-1))^{-\gamma}]$$

- With Market Clearing for α, FOC becomes

$$0 = \mu_R - r - \gamma\sigma_R^2 + \frac{1-\gamma}{\psi-1}\sigma_R\sigma_V + \lambda E[(e^{J_t}-1)e^{-\gamma J_t}]$$

Solution of Portfolio Problem

- Define

$$\mathrm{m}(\theta) = E[e^{\theta J_t} - 1]$$

- After some manipulations, we obtain the following system for risk premium, risk free rate, and PDE for V

$$E[\frac{dR}{R}] - r = (\gamma\sigma_C + \frac{\gamma\psi - 1}{\psi - 1}\sigma_V)(\sigma_c + \sigma_V) + \lambda \mathrm{m}(1) + \lambda \mathrm{m}(-\gamma) - \lambda \mathrm{m}(1-\gamma)$$

$$r = \rho + \frac{\mu_C}{\psi} - \frac{1 + \frac{1}{\psi}}{2}\gamma\sigma_C^2 - \frac{\gamma\psi - 1}{\psi - 1}\sigma_C\sigma_V - \frac{\gamma\psi - 1}{2(\psi - 1)}\sigma_V^2 - \lambda \mathrm{m}(-\gamma) + \frac{\frac{1}{\psi} - \gamma}{1 - \gamma}\lambda \mathrm{m}(1-\gamma)$$

$$0 = \frac{1}{V} + \mu_V + \mu_C + \sigma_C\sigma_V + \lambda \mathrm{m}(1) - E[\frac{dR}{R}]$$

As usual, this system of three equations gives a PDE for V

Case of Constant Intensity

- Let us examine stationary case, i.e. λ constant. Setup is similar to Martin "Consumption-Based Asset Pricing with Higher Cumulants"
- The system of equations become

$$E[\frac{dR}{R}] - r = \underbrace{\gamma\sigma_C^2}_{\text{standard model}} + \underbrace{\lambda \mathrm{m}(1) + \lambda \mathrm{m}(-\gamma) - \lambda \mathrm{m}(1-\gamma)}_{\text{disaster}}$$

$$r = \underbrace{\rho + \frac{\mu_C}{\psi} - \frac{1+\frac{1}{\psi}}{2}\gamma\sigma_C^2}_{\text{standard model}} \underbrace{- \lambda \mathrm{m}(-\gamma) + \frac{\frac{1}{\psi}-\gamma}{1-\gamma}\lambda \mathrm{m}(1-\gamma)}_{\text{disaster}}$$

$$0 = \frac{1}{V} + \lambda \mathrm{m}(1) - E[\frac{dR}{R}]$$

- Combining these equations,

$$V = \frac{1}{\rho - (1-\frac{1}{\psi})(\mu_C - \frac{\gamma}{2}\sigma_C^2 + \frac{1}{1-\gamma}\lambda \mathrm{m}(1-\gamma))}$$

Case of Time Varying Intensity

- Now let us have time varying intensity

$$\frac{dC_t}{C_{t\text{-}}} = \mu dt + \sigma dZ_t^C + (e^{J_t} - 1)dN_t$$
$$d\lambda_t = -\theta_\lambda(\lambda_t - \overline{\lambda}) + \nu_\lambda\sqrt{\lambda}dZ_t^\lambda$$

with $corr(dZ_t^\lambda, dZ_t^C) = 0$.
Setup from Wachter "Can Time-Varying Risk of Rare Disasters Explain Aggregate Stock Market Volatility?"

- V is only a function of $\lambda \Rightarrow$ only moves with dZ_t^λ, not with dZ_t^C, i.e.

$$\frac{dV}{V} = \mu_V dt + \sigma_V dZ_t^\lambda$$

- The system of equations becomes

$$E[\frac{dR}{R}] - r = \underbrace{\gamma\sigma_C^2}_{\text{standard model}} \underbrace{+\lambda \text{m}(1) + \lambda \text{m}(-\gamma) - \lambda \text{m}(1-\gamma)}_{\text{static disaster}} + \underbrace{\frac{\gamma\psi - 1}{\psi - 1}\sigma_V^2}_{\text{time-varying intensity}}$$

$$r = \underbrace{\rho + \frac{\mu_C}{\psi} - \frac{1+\frac{1}{\psi}}{2}\gamma\sigma_C^2}_{\text{standard model}} \underbrace{-\lambda \text{m}(-\gamma) + \frac{\frac{1}{\psi} - \gamma}{1-\gamma}\lambda \text{m}(1-\gamma)}_{\text{static disaster}} - \underbrace{\frac{\gamma\psi - 1}{2(\psi - 1)}\sigma_V^2}_{\text{time varying intensity}}$$

$$0 = \frac{1}{V} + \mu_V + \mu_C + \lambda \text{m}(1) - E[\frac{dR}{R}]$$

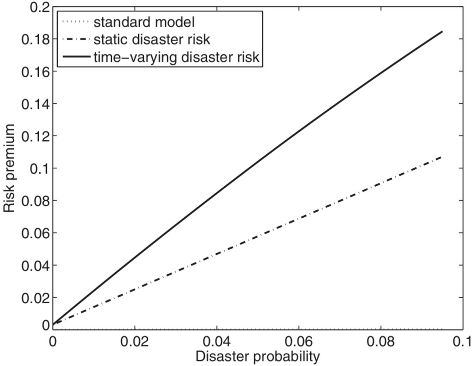
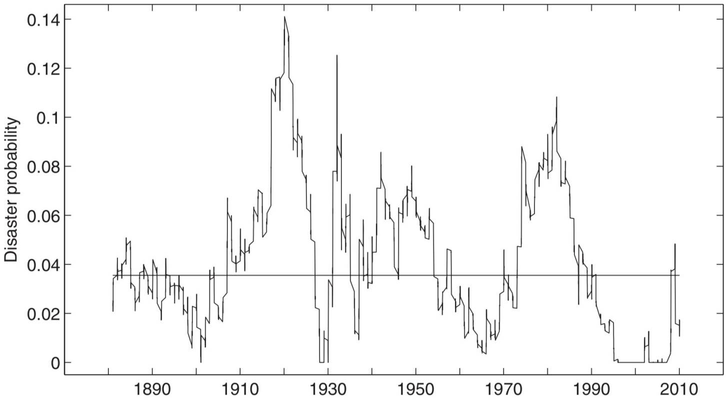


Figure 3. Decomposition of the equity premium in the time-varying disaster risk model. The solid line shows the instantaneous equity premium (the expected excess return on equity less the expected return on the government note), the dashed line shows the equity premium in a static model with disaster risk, and the dotted line shows what the equity premium would be if disaster risk were zero.


Disaster probability
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
1890
1910
1930
1950
1970
1990
2010

Empirical Test

Empirical Support: Barro (2006), Barro Ursua (2009)

- Are there disasters? Barro (2006)
- Do prices decline when probability of disaster increase? Barro and Ursua (2009)
    1. P(Consumption Drops by 10% | No Drop in Asset prices) = 1%
    2. P(Consumption Drops by 10% | Drop in Asset prices) = 3.8%

Critiques: Steinsson Nakamura (2013)

- Barro 2006 measures disaster from peak to through drop but typically disaster occurs across a few years, so annual drop lower
- Model supposes that drop of consumption is permanent, i.e. remember

$$\frac{dC_t}{C_{t\text{-}}} = \mu dt + \sigma dZ_t^C + (e^{J_t} - 1)dN_t$$

The may overstate the riskiness of consumption by failing to incorporate recoveries after disaster: Gourio 2008, Steinsson Nakamura 2013