Euler Equation in Continuous Time

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Continuous Time Mathematics

- Consider the discrete time process

$$Z_{t+\Delta t} = Z_t + \sqrt{\Delta t} \times \epsilon_t,\ \epsilon_t \sim N(0,1)$$

  When $\Delta \to 0$, the process has a limit called a **standard brownian motion**
- Formally, a standard brownian motion Z_t, $t \in [0, \infty)$ is a process such that:

$$Z_{t+s} - Z_t \sim \mathcal{N}(0, s)$$

Markovian Diffusion Process

- Consider the discrete time process

$$x_{t+\Delta t} = x_t + \mu(x_t)\Delta t + \sigma(x_t)\sqrt{\Delta t} \times \epsilon_t \; \epsilon_t \sim N(0,1)$$

  i.e.

$$E_t[x_{t+\Delta t} - x_t] = \mu(x_t)\Delta t$$
$$Var_t[x_{t+\Delta t} - x_t] = \sigma(x_t)^2 \Delta t$$

- A Markovian Diffusion process is the limit of this process as $\Delta t \to 0$. We formally write

$$dx_t = \mu(x_t)dt + \sigma(x_t)dZ_t$$

  where Z_t is a standard brownian motion.

- $\mu(x_t)$ is called the instantaneous drift and $\sigma(x_t)$ is called the instantaneous volatility. We can write

$$E_t[dx_t] = \mu(x_t)dt$$
$$Var_t[dx_t] = \sigma(x_t)^2 dt$$

Examples

- An **arithmetic** brownian motion is a process $x_t, t \in [0, \infty)$ such that

$$dx_t = \mu dt + \sigma dZ_t$$

  where Z_t is a standard brownian motion

- A **geometric** brownian motion is a process $x_t, t \in [0, \infty)$ such that

$$dx_t = \mu x_t dt + \sigma x_t dZ_t$$

  where Z_t is a standard brownian motion

Ito lemma

- Ito lemma. Given a process

$$dx_t = \mu(x_t)dt + \sigma(x_t)dZ_t$$

  We have

$$df(x_t) = f'(x_t)dx_t + \frac{1}{2}f''(x_t)\sigma^2(x_t)dt$$

- Heuristic derivation
  - Taylor expansion

$$f(x_{t+\Delta t}) - f(x_t) \approx f'(x_t)(x_{t+\Delta t} - x_t) + \frac{1}{2}f''(x_t)(x_{t+\Delta t} - x_t)^2 + \dots$$

  - Remember that

$$E[(x_{t+\Delta t} - x_t)^2] = \sigma(x_t)^2\Delta t + o(\Delta t)$$

    The important point is that the order of $E[x_{t+\Delta t} - x_t)^2]$ is Δt, not $(\Delta t)^2$.
  - Therefore, in expectation

$$E_t[f(x_{t+\Delta t}) - f(x_t)] = f'(x_t)E[x_{t+\Delta t} - x_t] + \frac{1}{2}f''(x_t)\sigma(x_t)^2\Delta t + o(\Delta t)$$

$$\Rightarrow E_t[df(x_t)] = f'(x_t)E[dx_t] + \frac{1}{2}f''(x_t)\sigma(x_t)^2 dt$$

Example

- Denote

$$\frac{dx_t}{x_t} = \mu dt + \sigma dZ_t$$

- μ_x is called the geometric drift and σ_x is called the geometric volatility Applying Ito's lemma on $\ln x$ gives

$$d \ln x_t = \frac{dx_t}{x_t} - \frac{1}{2}\sigma dt$$

Case of Multiple Shocks

- Consider the following system

$$dx_t = \mu_1(x_t)dt + \sigma_1(x_t)dZ_1 + \cdots + \sigma_m(x_t)dZ_m$$

where $dZ_{1t}, \ldots dZ_{mt}$ are uncorrelated shocks

- It can be written in vector form:

$$dx_t = \mu(x_t)dt + \sigma(x_t)' dZ_t$$

where $\sigma(x_t), dZ_t$ are vectors of size $m \times 1$

- Consider a function $f : \mathbb{R} \longrightarrow \mathbb{R}$. Ito's lemma is

$$df(x_t) = f'(x_t)dx + \frac{1}{2}f''(x_t)\sigma'(x_t)\sigma(x_t)dt$$

Case of Multiple Variables

- Consider the following system

$$dx_1 = \mu_1(x_1, \ldots, x_n)dt + \sigma_1(x_1, \ldots, x_n)dZ_t$$

$$\vdots$$

$$dx_n = \mu_n(x_1, \ldots, x_n)dt + \sigma_n(x_1, \ldots, x_n)dZ_t$$

- It can be written in vector form:

$$d\mathbf{x}_t = \boldsymbol{\mu}(\mathbf{x})dt + \boldsymbol{\sigma}(x)dZ_t$$

where $\mathbf{x}$, $\boldsymbol{\mu}(\mathbf{x})$, and $\boldsymbol{\sigma}(\mathbf{x})$ are vectors of size $n \times 1$

- Consider a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$. Ito's lemma is

$$df(\mathbf{x}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial f}{\partial x_i \partial x_j} \sigma_i \sigma_j dt$$

Example

Denote

$$\frac{dx}{x} = \mu_x dt + \sigma_x dZ_t$$
$$\frac{dy}{y} = \mu_y dt + \sigma_y dZ_t$$

Applying Ito's lemma on $x^\alpha y^\beta$ gives

$$\frac{d(x^\alpha y^\beta)}{x^\alpha y^\beta} = \alpha\frac{dx}{x} + \beta\frac{dy}{y} + \frac{1}{2}\alpha(\alpha-1)\sigma_x^2 dt + \frac{1}{2}\beta(\beta-1)\sigma_y^2 dt + \alpha\beta\sigma_x\sigma_y dt$$

- Special case $\alpha = 1$ and $\beta = 1$

$$\frac{d(xy)}{xy} = \frac{dx}{x} + \frac{dy}{y} + \sigma_x\sigma_y dt$$

- Special case $\alpha = 1$ and $\beta = -1$

$$\frac{d(x/y)}{x/y} = \frac{dx}{x} - \frac{dy}{y} + \sigma_y^2 dt - \sigma_x\sigma_y dt$$

SDF in Continuous Time

SDF in Continuous Time

- The stochastic discount factor is defined as the process Λ_t such that

$$\Lambda_t P_t = E_t \int_t^{+\infty} \Lambda_s D_s ds$$

- For a short period of time Δt

$$\Lambda_t P_t = \Lambda_t D_t \Delta t + E_t[\Lambda_{t+\Delta t} P_{t+\Delta t}]$$
$$\Rightarrow 0 = \Lambda_t D_t dt + E[d(\Lambda_t P_t)]$$

Dividing by $\Lambda_t P_t$, we obtain

$$0 = \frac{D_t}{P_t} dt + E[\frac{d(\Lambda_t P_t)}{\Lambda_t P_t}] \quad (1)$$

Applying Ito's Lemma we obtain

$$0 = \frac{D_t}{P_t} + \mu_{\Lambda t} + \mu_{Pt} + \sigma_{Pt}\sigma_{\Lambda t} \quad (2)$$

where μ_{Pt} and σ_{Pt} denote the geometric drift and volatility of P_t, and $\mu_{\Lambda t}$ and $\sigma_{\Lambda t}$ denotes the geometric drift and volatility of Λ_t

SDF in Continuous Time

- The instantaneous return is defined as

$$dR_t = \frac{D_t dt + dP_t}{P_t}$$

- Plugging into Equation (2), we obtain

$$\mu_{Rt} = -\mu_{\Lambda t} - \sigma_{\Lambda t}\sigma_{Rt}$$

where μ_{Rt} and σ_{Rt} denote the drift and volatility of dR_t

Discrete vs Continuous Time

Discrete Time	Continuous time
$P_t = E_t \sum_{t+1}^{+\infty} m_{t,t+\tau} D_\tau$	$P_t = E_t \int_t^{+\infty} \frac{\Lambda_\tau}{\Lambda_t} D_\tau d\tau$
$R_{t+1} = \frac{D_{t+1}+P_{t+1}}{P_t}$	$dR_t = \frac{D_t dt + dP_t}{P_t}$
$E[R_{it+1}] = \frac{1}{E_t[m_{t,t+1}]} - cov_t(\frac{m_{t,t+1}}{E_t[m_{t,t+1}]}, R_{it+1})$	$\mu_{Rt} = -\mu_{\Lambda t} - \sigma_{\Lambda t}\sigma_{Rt}$

- Applying equation Equation (2) with instantaneous risk free rate r_t we obtain

$$r_t = -\mu_{\Lambda t}$$

- We can rewrite Equation (2) as

$$\mu_{Rt} - r_t = -\sigma_{\Lambda t}\sigma_{Rt}$$

$-\sigma_{\Lambda_t}$ is called the market price of risk, and is often denoted by κ_t

Portfolio Problem

Asset Returns

- Assume that the household can trade a risk free asset r_t and a risky asset with a return dR_t given by

$$dR_t = (r_t + \sigma_{Rt}\kappa_t)dt + \sigma_{Rt}dZ_t$$

- The investor chooses a consumption C_t and a share α_t of wealth invested in risky assets. Given this choice, the wealth of the investor evolves as

$$\begin{aligned} dW_t &= W_t\alpha_t dR_t + W_t(1-\alpha_t)r_t dt - C_t dt \\ \Rightarrow dW_t &= r_t W_t dt + W_t\alpha_t(dR_t - r_t dt) - C_t dt \\ \Rightarrow dW_t &= (r_t W_t + W_t\alpha_t\sigma_{Rt}\kappa_t - C_t)dt + W_t\alpha_t\sigma_{Rt}dZ_t \end{aligned}$$

- The investor problem is

$$J_t = \max_{C,\sigma_W} \boldsymbol{E}_t \left\{ \int_t^{+\infty} e^{-\rho(\tau - t)} U(C_\tau) d\tau \right\}$$
$$W_0 > 0$$
$$W_t > 0$$
$$C_t > 0$$

- For a short period of time Δt:

$$J_t = \max_{C_t,\alpha_t} \left\{ U(C_t)\Delta t + e^{-\rho\Delta t} E_t[J_{t+\Delta t}] \right\}$$
$$\Rightarrow 0 = \max_{C_t,\alpha_t} \left\{ U(C_t)\Delta t + (1 - \rho\Delta t)E[J_{t+\Delta t}] - J_t + o(\Delta t) \right\}$$
$$\Rightarrow 0 = \max_{C_t,\alpha_t} \left\{ U(C_t)\Delta t + E[J_{t+\Delta t} - J_t] - \rho J_t \Delta t + o(\Delta t) \right\}$$
$$\Rightarrow 0 = \max_{C_t,\alpha_t} \left\{ U(C_t)dt + E_t[dJ_t] - \rho J_t dt \right\}$$

This is called "Hamilton Jacobi Bellman" equation

CRRA utilities.

- From now on, assume CRRA utilities

$$U(C) = \frac{C^{1-\gamma}}{1-\gamma}$$

- In this case, guess that the value function is

$$J_t = \frac{(W_t\xi_t)^{1-\gamma}}{1-\gamma}$$

  ξ can be seen as a networth multiplier
- The HJB equation can be written as

$$0 = \max_{C_t,\alpha_t} \left\{ \frac{C_t^{1-\gamma}}{1-\gamma} + E[\frac{d(W_t\xi_t)^{1-\gamma}}{1-\gamma}] - \rho \frac{(W_t\xi_t)^{1-\gamma}}{1-\gamma} \right\}$$

- Dividing by the value function $\frac{(W_t\xi_t)^{1-\gamma}}{1-\gamma}$ and applying Ito's Lemma

$$0 = \max_{C_t,\alpha_t} \left\{ \frac{C_t^{1-\gamma}}{W_t^{1-\gamma}\xi_t^{1-\gamma}} + (1-\gamma)\mu_{Wt} + (1-\gamma)\mu_{\xi t} - \frac{(1-\gamma)\gamma}{2}\sigma_{Wt}^2 - \frac{(1-\gamma)\gamma}{2}\sigma_{\xi t}^2 + (1-\gamma)^2\sigma_{Wt}\sigma_{\xi t} - \rho \right\}$$

  where μ_{Wt}, α_t denote geometric drift and volatility of wealth, and $\mu_{\xi t}, \sigma_{\xi t}$ denote geometric drift and volatility of ξ_t

FOC

- Substituting the law of motion of wealth:

$$0 = \max_{C_t,\alpha_t} \left\{ \frac{C_t^{1-\gamma}}{W_t^{1-\gamma}\xi_t^{1-\gamma}} + (1-\gamma)(r_t + \kappa_t\alpha_t\sigma_{Rt} - \frac{C_t}{W_t}) + (1-\gamma)\mu_{\xi t} - \frac{(1-\gamma)\gamma}{2}\alpha_t^2\sigma_{Rt}^2 - \frac{(1-\gamma)\gamma}{2}\sigma_{\xi t}^2 + (1-\gamma)^2\alpha_t\sigma_{Rt}\sigma_{\xi t} - \rho \right\}$$

- The FOCs are:

$$[C] : C_t = \xi_t^{1-\frac{1}{\gamma}} W_t \tag{3}$$

$$[\alpha] : \alpha = \underbrace{\frac{1}{\gamma}\frac{\kappa_t}{\sigma_{Rt}}}_{\text{myopic demand}} + \underbrace{\frac{1-\gamma}{\gamma}\frac{\sigma_{\xi t}}{\sigma_{Rt}}}_{\text{int. hedging demand}} \tag{4}$$

CAPM

- Inverting the FOC for α_t, we can write κ_t as:

$$\begin{aligned}\kappa_t &= \gamma \alpha_t \sigma_{Rt} - (1-\gamma)\sigma_{\xi t} \\ &= \gamma \sigma_{Wt} - (1-\gamma)\sigma_{\xi t}\end{aligned}$$

- Using the definition of κ

$$\mu_R - r = \underbrace{\gamma \sigma_{Rt} \sigma_{Wt}}_{\text{myopic demand}} + \underbrace{(\gamma - 1)\sigma_{Rt}\sigma_{\xi t}}_{\text{intertemporal hedging demand}}$$

This relates the return of any asset to its covariance with wealth and investment opportunity (ICAPM)

Discrete vs Continuous Time

Discrete Time	Continuous time
$\mathbb{E}_t r_{t+1} + \frac{\sigma_r^2}{2} - r_{t+1}^f = \gamma\sigma_{rw} + (\gamma-1)\sigma_{r,(E_{t+1}-E_t)\sum_{t=1}^{+\infty}\rho_j r_{w,t+1+j}}$	$\mu_R - r = \gamma\sigma_R\sigma_W + (\gamma-1)\sigma_R\sigma_\xi$

ODE for ξ_t

- Plugging FOC for C in HJB, we can express ξ_t in a forward looking way:

$$\frac{C_t}{W_t} = \xi_t^{1-\frac{1}{\gamma}}$$

$$= \frac{\rho}{\gamma} + (1 - \frac{1}{\gamma}) \left(\underbrace{r_t + \sigma_{Wt}\kappa_t}_{\text{port. return}} + \underbrace{\mu_{\xi t}}_{\text{change in inv. opp.}} - \underbrace{\frac{\gamma}{2}(\sigma_{Wt}^2 + \sigma_{\xi t}^2 - 2\frac{1-\gamma}{\gamma}\sigma_{Wt}\sigma_{\xi t})}_{\text{precautionary savings}} \right)$$

When r or κ is high, ξ is high (ξ proxies for future investment opportunities)

Euler Equation

- Applying Ito's lemma on $C_t = \xi_t^{1-\frac{1}{\gamma}} W_t$, we get

$$\sigma_{Ct} = \sigma_{Wt} + \frac{\gamma - 1}{\gamma}\sigma_{\xi t}$$

$$\mu_{Ct} = \mu_{Wt} + \frac{\gamma - 1}{\gamma} - \frac{\gamma - 1}{2\gamma^2}\sigma_{\xi t}^2 + \frac{\gamma - 1}{\gamma}\sigma_{Wt}\sigma_{\xi t}$$

where μ_{Ct}, σ_{Ct} denote the geometric drift and volatility of consumption

- Substituting σ_{Wt} using FOC for α_t and μ_{Wt} using law of motion of wealth , we obtain the Euler Equation

$$\sigma_{Ct} = \frac{\kappa_t}{\gamma}$$

$$\mu_{Ct} = \frac{r_t - \rho}{\gamma} + \frac{1 + \gamma}{2\gamma^2}\kappa_t^2$$

- This can also be written as

$$\kappa_t = \gamma\sigma_{Ct}$$

$$r_t = \rho + \gamma\mu_{Ct} - \frac{(1 + \gamma)\gamma}{2}\sigma_{Ct}^2$$

Discrete vs Continuous Time

Discrete Time	Continuous time
$\sum_0^{+\infty} \beta^t U(C_t)$	$\int_0^{+\infty} e^{-\rho t} U(C_t) dt$
$\Delta c_{t+1} \sim N(\mu_c, \sigma_c)$	$\frac{dC}{C} = \mu_C dt + \sigma_C dZ_t$
$r^f_{t+1} = -\log \beta + \gamma \mu_c - \frac{\gamma^2}{2} \sigma_c^2$	$r = \rho + \gamma \mu_C - \frac{(1+\gamma)\gamma}{2} \sigma_C^2$
$\mathbb{E}_t r_{t+1} + \frac{\sigma_r^2}{2} - r^f_{t+1} = \gamma \sigma_{rc}$	$\mu_R - r = \gamma \sigma_R \sigma_C$