Asset Prices and Wealth Inequality*

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Abstract

I use recently available data on the top of the wealth distribution to study the relationship between wealth inequality and asset prices. I document two stylized facts: (1) when stock market returns are high, wealth inequality increases (2) higher wealth inequality predicts lower future excess stock returns. These facts correspond to the basic predictions of an asset pricing model where agents have heterogeneous preferences. Quantitatively, however, a model that matches the volatility of asset prices tends to imply a wealth distribution with a right tail thicker than the data. I suggest two parsimonious deviations to resolve this tension: (i) “live-fast-die-young dynamics”, in which risk-tolerant investors remain levered only for a short period of time, or (ii) time-varying investment opportunities for rich households compared to the rest of the distribution.

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1 Introduction

Recent empirical studies have documented important fluctuations in wealth inequality over the past century.\footnote{See, for instance, Kopczuk and Saez (2004), Piketty (2014), and Saez and Zucman (2016).} Volatile stock market returns potentially account for these fluctuations. Conversely, a large theoretical literature in asset pricing examines the role of household heterogeneity in shaping asset prices, but seldom considers its implication on wealth inequality. In this paper, I use this recently available data on wealth inequality to examine empirically and theoretically the relationship between asset prices and the wealth distribution.

I focus on the following mechanism. Risk-tolerant investors hold more risky assets, accumulate more wealth, and disproportionately end up at the top of the wealth distribution. As a consequence, in periods when stocks enjoy large realized returns, investors at the top of the wealth distribution gain more than the rest, i.e. wealth inequality increases. In turn, as a larger share of wealth is owned by risk-tolerant households, aggregate demand for risk increases, which lowers risk premia and pushes up asset prices, i.e. higher wealth inequality predicts lower future excess returns. I confirm empirically this joint dynamic between asset prices and wealth inequality. In response to a realized stock return of 10\%, the wealth share of the top 0.01\% increases by 4.5\%. In turn, a 10\% increase in the wealth share of the top 0.01\% predicts lower future excess returns by one percentage point over the next year.

I then evaluate whether this mechanism can quantitatively explain the excess volatility of asset prices in equilibrium. I use the reduced-form evidence I documented earlier to estimate a state-of-the-art asset pricing model with heterogeneous agents. I find that there is a key tension between the model to the data: a model that match the excess volatility of asset prices requires such a large degree of preference heterogeneity that it gives rise to a wealth distribution close to Zipf’s law, i.e. with a right tail much thicker than the data. To solve this tension, I propose two possible parsimonious deviations from the model: (i) “live-fast-die-young dynamics”, in which risk-tolerant investors remain highly levered only for a short period of time, or (ii) time-varying investment opportunities for rich households compared to the rest of the distribution.

The paper proceeds in four stages. First, I present two stylized facts on the relationship between asset prices and the wealth distribution. I first show that, because top households tend to hold more equity, stock market returns generate large fluctuations in wealth inequality. To show this, I use top the series of top wealth shares constructed from tax filings by Kopczuk and Saez (2004)
and from Forbes 400 to estimate the exposure of the top percentiles to stock market returns. In response to a realized stock return of 10%, the average wealth increases by 5%, while the average wealth for the top 0.01% increases by 9.5% (the wealth share of the top 0.01% increases by the difference of 4.5%). The flip side of this relationship is that, in an economy where inequality is high, the share of wealth owned by risk-tolerant investors is high, and therefore, in equilibrium, risk premia are low. Thus, higher inequality should predict lower future returns. Indeed, in the data, I find that a 10% increase in the wealth share of the top 0.01% predicts lower future excess returns by one percentage points over the following year.

Second, I examine those facts through the lens of an asset pricing model with heterogeneous agents. Specifically, I study the joint dynamics of asset prices and wealth inequality in a continuous-time, overlapping generations framework where agents differ with respect to their relative risk aversion (RRA) and elasticity of intertemporal substitution (EIS), following Gărlăneanu and Panageas (2015). The model can qualitatively generate my two stylized facts. One key contribution of my paper is to study the wealth distribution in the model. I show that the wealth distribution in this model exhibits a thick right tail, as in the data. The thickness of its tail is determined by the average logarithmic wealth growth rate of top households relative to the economy.

Third, I ask whether the model can explain the excess volatility of asset prices in equilibrium. I find that, quantitatively, there is tension between the model and the data. To generate asset prices as volatile as the data, the model requires a large degree of preference heterogeneity. In turn, this generates a wealth distribution with a right tail that is too thick compared to the data. This tension arises independently of the source of preference heterogeneity: it is present whether households differ with respect to their relative risk aversion, intertemporal elasticity of substitution, or/and subjective discount factors. Allowing for heterogeneous labor income or idiosyncratic shocks, which are realistic ingredients for the wealth distribution, does not help resolve the tension.

Fourth, I propose two parsimonious deviations from the baseline model that help resolve this tension. The first departure is to assume that risk-tolerant investors become risk-averse after a short amount of time. This makes the right tail of the wealth distribution thinner compared to the baseline model, thereby resolving the tension described above. This departure is consistent with the fact that older households in top percentiles tend to be less levered than younger households in top percentiles. The second departure is time-varying investment opportunities for risk-tolerant households relative to risk-averse households. These shocks generate additional, low-frequency changes in wealth inequality, which increases the excess volatility of asset prices without increasing
the long-run level of inequality. Therefore, these shocks help the model to better fit the data. Moreover, they can explain certain dynamics in wealth inequality that cannot be fully accounted by stock market returns, in particular the decrease of wealth inequality after the Great Depression, or the rise in wealth inequality at the end of the 20th century.

**Related Literature.** This paper contributes to the growing literature on wealth inequality. On the empirical side, I rely critically on the recent wealth shares constructed by Kopczuk and Saez (2004). On the theoretical side, random growth theories of the wealth distribution include Wold and Whittle (1957), and, more recently, Benhabib et al. (2011), Benhabib et al. (2015b), Benhabib et al. (2016), Jones (2015), and Cao and Luo (2016). While a number of studies focus on the role of the risk-free rate of return in shaping the wealth distribution (Piketty (2014), Acemoglu and Robinson (2015)), I document a more important role for the rate of return on risky assets. One key contribution relative to this literature is to study an economy in which the dynamics of the wealth distribution evolves over time. I show that, in this set up, the tail index of the wealth distribution is determined by the average logarithmic wealth growth rate of top households relative to the economy.\(^2\) This extends the tools developed by Luttmer (2012) and Gabaix et al. (2016), who examine the transition of a distribution between two steady states, to a general economy with a Markovian structure.\(^3\)

I study the effect of wealth inequality for asset prices through the lens of an asset pricing model where agents have heterogeneous agents. Therefore, this paper contributes to a large asset pricing literature where agents have heterogeneous preferences.\(^4\) As pointed out by Dumas (1989), preference heterogeneity tends to imply exploding wealth distribution in infinite-horizon economies. To solve this issue, Chan and Kogan (2002) assumes “catching up with the Joneses” preferences, which means that all investors grow at the same average rate. While this makes the model particularly tractable, it also means that risk tolerant investors do not disproportionately end up at the top of the distribution. Therefore, the model cannot generate the two-way feedback between wealth inequality and asset prices that I observe in the data. Guvenen (2009) presents a model that combines

\(^2\)The role of this object in shaping the wealth distribution ties this paper to the literature on growth-optimal portfolios, which is initiated by Kelly (1956). See also see Blume et al. (1992), or, more recently, Borovička (Forthcoming).

\(^3\)As in these papers, heterogeneity across agents is key to obtain fast transition dynamics. More precisely, a model with preference heterogeneity is a particular case of the “type-dependence” model explored in Gabaix et al. (2016).

\(^4\)The literature is too large to be summarized exhaustively, so I focus on few models trying to match quantitatively asset prices in the U.S.
preference heterogeneity with limited participation, where heterogeneity in EIS is key to generate volatile asset prices. However, to obtain a stationary wealth distribution, the model has to abstract from long-run growth, which makes it difficult to take it to the data. One key difference with my paper is that I focus on the heterogeneity within stock-holders, rather than between stockholders and non-stockholders. Most related to this paper, Gărleanu and Panageas (2015) studies the role of preference heterogeneity in an overlapping generation model. Because investors die after a certain time, the model naturally gives rise to a stationary wealth distribution. A key contribution of my paper is to show that preference heterogeneity still drives the tail index of the wealth distribution. After estimating the model on the data on wealth inequality, I show that the model cannot match asset prices without implying a wealth distribution close to Zipf’s law, i.e. with a right tail too thick compared to the data. I suggest two parsimonious deviations to resolve this tension.

A large literature in household finance examines the heterogeneity in portfolio choice across the wealth distribution (Guiso et al. (1996), Carroll (2000), Campbell (2006), Wachter and Yogo (2010), Roussanov (2010), Calvet and Sodini (2014), and Fagereng et al. (2016)). Parker and Vissing-Jørgensen (2009) and Guvenen et al. (2017) study the exposure of the labor income of top households to aggregate shocks. Malloy et al. (2009) documents that the consumption of richer stockholders tends to be more exposed to stock market returns. Most related to this paper, Bach et al. (2015) stresses the heterogeneous exposure to aggregate risk at the top of the wealth distribution using Swedish data. A growing literature studies the impact of investment heterogeneity on wealth inequality and asset prices. Gollier (2001) is an early example that examines theoretically the importance of the wealth distribution for asset prices. Barczyk and Kredler (2016) examines theoretically the role of inequality and incomplete markets on asset prices. Eisfeldt et al. (2016) examines the joint relation between the wealth distribution and asset prices across markets with different expertises. On a more empirical side, Johnson (2012) examines the role of income inequality shocks for the cross-section of returns, Favilukis (2013) examines the role of changes in participation cost and wage inequality on asset prices. Kacperczyk et al. (2018) studies the role of investor sophistication for the recent rise in capital income inequality. Most relevant to this paper, Toda and Walsh (2016) independently shows that fluctuations in income inequality negatively predict future excess stock returns, using the series on top income shares from Piketty and Saez (2003).

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5 This paper focuses on the role of aggregate shocks for the wealth distribution. Other papers consider the role of displacement shocks (Gărleanu et al. (2012), Gărleanu and Panageas (2017)), idiosyncratic shocks (Constantinides and Duffie (1996), Storesletten et al. (2007), Kogan et al. (Forthcoming), Schmidt (2016)), fluctuating capital shares (Lettau et al. (2016), Greenwald et al. (2014)) or fluctuating tax rates (Pastor and Veronesi (2016)).
My paper confirms these results using the series of top wealth shares shares from Kopczuk and Saez (2004). Relative to this paper, I examine the magnitude of this effect through a quantitative model. This leads me to highlight some other moments, more precisely estimated, that discipline the effect wealth inequality can have on asset prices (e.g. the exposure of top households to to stock market returns and the tail index of the wealth distribution).

**Road Map** The rest of my paper is organized as follows. In Section 2, I document two key stylized facts about the relation between wealth inequality and asset prices. In Section 3, I present a standard asset pricing model with heterogeneous agents to interpret these findings. In Section 4, I characterize analytically the wealth distribution in the model. In Section 5, I highlight the key tension to match quantitatively the model to the data. In Section 6, I propose two parsimonious deviations that help resolve the tension. Section 7 concludes.

## 2 Data and Facts

I now analyze data about the top of the wealth distribution to document two stylized facts predicted by heterogeneous agents models. In particular, I focus on the following mechanism. Risk-tolerant households invest more in risky assets and disproportionately end up at the top of the wealth distribution. In periods when stocks enjoy large realized returns, investors at the top of the wealth distribution gain more than the rest; thus, inequality increases. In turn, as a larger share of wealth falls into the hands of risk-tolerant households, the aggregate demand for risk increases, which lowers risk premia; thus, higher inequality predicts lower future returns. After introducing the data, I document facts reflecting each step of this mechanism.

### 2.1 Data

**Wealth Shares** I am interested in measuring changes in the wealth distribution and their relationship to stock returns. Therefore, I need yearly estimates of the wealth distribution that cover several business cycles. I use two datasets that, together, cover most of the last 100 years.

The first wealth series is the annual series of top wealth shares constructed by Kopczuk and Saez (2004). This series is constructed from estate tax returns, which report the wealth of deceased individuals, above a certain wealth threshold. From the wealth distribution of the deceased, Kopczuk and Saez (2004) estimates the wealth distribution of the living using the mortality mul-

Another data series about wealth inequality is Saez and Zucman (2016). They construct top wealth shares from income tax returns, using a capitalization method. However, the series builds in smoothing over time, which makes it harder to examine the joint dynamics of asset prices and top wealth shares. I compare the two series more thoroughly in Appendix A.

I supplement the series of top wealth shares with the list of the wealthiest 400 Americans constructed by Forbes Magazine every year since 1982, which offers an unparalleled view on the right tail of the wealth distribution. The list is created by a dedicated staff of the magazine, based on a mix of public and private information. The total wealth of individuals on the list accounts for approximately 1.5% of total aggregate wealth in 2010.

Asset prices For asset prices, I use yearly stock market returns and risk-free rates from Shiller (2015). The excess stock market return is measured as the log stock market return minus the log risk-free rate. Data on the price-dividend ratio and the price-payout ratio comes from Welch and Goyal (2008).

2.2 Wealth Exposure to the Stock Market Across the Wealth Distribution

The basic building block of heterogeneous agents models is that there is a group of investors that is disproportionately exposed to aggregate shock. Following a positive return, these households gain more relative to other households; therefore, the wealth distribution fluctuates.

6Forbes Magazine reports: “We pored over hundreds of Securities Exchange Commission documents, court records, probate records, federal financial disclosures and Web and print stories. We took into account all assets: stakes in public and private companies, real estate, art, yachts, planes, ranches, vineyards, jewelry, car collections and more. We also factored in debt. Of course, we don’t pretend to know what is listed on each billionaire’s private balance sheet, although some candidates do provide paperwork to that effect.”

7Recent empirical studies examining the Forbes 400 list also include Klass et al. (2006) and Kaplan and Rauh (2013).

To measure the heterogeneity in risk exposure across the wealth distribution, I estimate the wealth exposure to stock market returns at different percentiles of the distribution. More precisely, I define the exposure of households in a given group as the slope estimate of a regression of the growth of total wealth in the group on excess stock market returns, i.e.,

\[
\log \left( \frac{W_{G,t-1+h}}{W_{G,t-1}} \right) - h \log R_{ft} = \alpha_{Gh} + \beta_{Gh} (\log R_{Mt} - \log R_{ft}) + \epsilon_{Ght}
\]

where \( W_{G,t} \) denotes the total wealth of households in the percentile group \( G \) in year \( t \), \( \log R_{Mt} \) denotes the log stock market return, and \( \log R_{ft} \) denotes the log risk-free rate. The dependent variable is the log ratio of wealth in the percentile group \( G \) from year \( t - 1 \) to year \( t - 1 + h \). Note that the starting year is \( t - 1 \): this is because \( W_{G,t} \) represents the average wealth owned by the group during year \( t \), rather than the wealth owned at the beginning of the year.

The first four columns in Table 1 (Panel A) report the estimates for \( \beta_{G4} \), the wealth exposure to the stock market at the four year horizon (\( h = 4 \)),\(^9\) for four groups of households: all households, households in the top 1 – 0.1%, households in the top 0.1% – 0.01%, and households in the top 0.01%. The estimated exposure \( \beta_{G4} \) increases with the top percentiles, from 0.48 for the average household, to 0.95 for households in the top 0.01%. The last column of Panel A in Table 1 reports the wealth exposure of the Top 400 from Forbes. The estimates for households in the extreme tail of the distribution are similar in magnitude to the estimates for households in the top 0.01% from tax data. This is reassuring, because these datasets are constructed from two completely different sources. In short, top households are twice as exposed to stock market returns as the representative household.

Since top households are comparatively more exposed to the stock market, high stock market returns increase inequality. Panel B of Table 1 confirms this relationship by regressing top wealth shares on stock market returns. The estimate 0.48 corresponds to the difference of exposure between households at the top and the average household (\( \approx 0.95 - 0.48 \)). Moreover, the regression shows that the difference is statistically significant.

**Horizon** A large share of wealth in top percentiles is held in privately-held assets, which are not typically traded. If the valuations of this non-traded wealth reacts more sluggishly to changes in the stock market, this introduces a bias in the estimate: top wealth shares may not react immediately to changes in stock market returns.\(^{10}\)

\(^9\)The choice for the 4-year horizon is justified in the next paragraph.

\(^{10}\)The valuation of this private wealth depends on the source. For estate tax, valuation of non-tradable assets is done by an external appraiser. Forbes magazine uses the valuation implied by the most recent financing round, or the
To handle this stale-pricing problem, I examine the reaction of top wealth shares at different horizons. Figure 1 plots the estimate of $\beta_h$ obtained by making the horizon $h$ varies from $h = 1$ to $h = 8$ in Equation (1). This traces the impulse response of top wealth shares to a stock market return shock. The plot shows that the estimates tends to increase from $h = 1$ to $h = 4$, consistent with the stale pricing problem. The estimate peaks at the four year horizon, justifying the choice of horizon $h = 4$ in the regressions above.12

Composition Changes Top percentiles do not necessary include the same individuals over time — some people enter and drop from the top every year. This may bias the regressions above. For instance, if there were more entrants in top percentiles when stock market returns are high, the exposure of top wealth shares to stock market returns would be higher than the actual wealth exposure of top households. To address this concern, I use the panel dimension of Forbes 400. Each year, I construct the average wealth growth of households in the Top 400, whether or not they drop from the list by the end of the year. This yearly series differs from the total growth of the top 400, by removing the effect of compositional changes on the growth of top wealth shares.13 Table 2 shows that I obtain very similar results with this new series, which means that compositional changes play no role in driving the stock market exposure of top percentiles. In Appendix A, I show that it comes from the fact that year-to-year changes in stock market returns are large compared to year-to-year changes in the idiosyncratic volatility of wealth.

2.3 Top Wealth Shares and Future Excess Returns

The previous evidence suggests that wealthy households are more willing to take on aggregate risk. The flip side of this relationship is that, as top wealth shares increase, wealth is rebalanced from risk-averse households to risk-tolerant households; therefore, the total demand for risk in the economy increases. In equilibrium, the compensation for holding risk decreases. Hence, higher top wealth shares should predict lower future excess returns.

I measure the predictive power of top wealth shares by regressing excess stock returns on the prevailing price-to-earnings ratios for similar public companies. This second method corresponds to the methodology of the Financial Accounts of the United States to estimate total wealth in private equity.

12Relatedly, Brav et al. (2002), Malloy et al. (2009) show that reported consumption growth for richer households is more correlated to aggregate consumption growth at longer horizon.
13See Gomez (2018) for more details about this construction.
wealth share of the top 0.01%:

\[
\sum_{1 \leq h \leq H} \log R_{M,t+h} - \log R_{f,t+h} = \alpha + \beta_H \log \text{Wealth Share Top 0.01\%}_t + \epsilon_{Ht} \tag{2}
\]

where \( h \) denotes the horizon, \( \log R_{M,t} \) denotes the log stock market return, and \( \log R_{f,t} \) denotes the log risk-free rate.

The first line in Table 3 reports the results of the predictability regression at the one-year and three-year horizons using the wealth share of the top 0.01% as a predictor. The estimate is negative and significant at the 10% level. Quantitatively, a 10% increase in the wealth share of the top 0.01% is associated with a decrease of excess returns by one percentage point over the next year.

It is well known that, for a predictor that is persistent and correlated with returns, like top wealth shares, conventional t-statistics are misleading.\(^{14}\) To address this concern, I rely on a test developed in Campbell and Yogo (2006), which is valid even when the predictor variable has a root close to or larger than one.\(^{15}\) As reported in Table 4, the hypothesis that the top wealth share has a unit-root cannot be rejected. Still, even after allowing for explosive dynamics in top wealth shares, the wealth share of the top 0.01% is found to significantly predict returns.

Another way to correct for the persistence of the predictor is to use a detrended version of the predictor (see Hodrick (1992)). Table 3 also reports the predictability regression using both the five-year difference and the detrended version of the top wealth share. The predictive power of top wealth shares remains significant.

Finally, I examine whether the information in the wealth share of the top 0.01% is subsumed by the price-dividend ratio, which is often used as a predictor of excess returns.\(^{16}\) I run the following bivariate predictive regression:

\[
\sum_{1 \leq h \leq H} \log R_{M,t+h} - \log R_{f,t+h} = \alpha + \beta_H \log \text{Wealth Share Top 0.01\%}_t + \gamma_H dp_t + \epsilon_{Ht} \tag{3}
\]

Table 3 reports that the predictive power of top wealth shares remains substantial after one adds the dividend price or the dividend payout as a predictor.

I have shown that fluctuations in stock prices generate fluctuations in inequality, and, in turn, that the level of inequality determines future excess returns. Those facts are at the heart of asset pricing models with heterogeneous agents. I now examine these facts within a quantitative model.

\(^{14}\)See, for instance, Elliott and Stock (1994) and Stambaugh (1999).

\(^{15}\)The test can only be done for the restricted sample without gaps in the predictor, i.e. 1917-1951.

\(^{16}\)Campbell and Shiller (1988)
3 Asset Pricing Model with Heterogeneous Preferences

I consider a continuous-time pure-exchange economy. I present a model where overlapping generations of households differ in their preferences, which follows Gärleanu and Panageas (2015). I derive the dynamics of individual wealth, as well as the dynamics of the asset prices in the model.\(^{17}\)

3.1 Setup

Endowment I consider a continuous-time pure exchange economy. The aggregate endowment per capita exhibits i.i.d. growth, i.e. its law of motion is

\[
\frac{dY_t}{Y_t} = \mu dt + \sigma dZ_t
\]  

where \(Z = \{Z_t \in \mathbb{R}|F_t, t \geq 0\}\) is a standard Brownian motion defined on a probability space \((\Omega, P, \mathcal{F})\), equipped with a filtration \(\mathcal{F} = \{\mathcal{F}_t, t \geq 0\}\) with the usual conditions.

Demographics The specification of demographics follows Blanchard (1985). Each agent faces a constant hazard rate of death \(\delta > 0\). Total population size, denoted, \(N_t\) grows at rate \(n\). During a short time period \(dt\), a mass \(\delta dt\) of the population dies and a new cohort of mass \((\delta + n)dt\) is born.

Labor Income. An agent \(i\) born at time \(s(i)\) is endowed with the labor income process \(L_i = \{L_{it}: t \geq s(i)\}\), given by

\[
L_{it} = \omega Y_t \times \chi_i \times G(t - s(i))
\]  

The first term of this formula, \(\omega Y_t\), corresponds to the fraction of the aggregate endowment distributed as labor income. The second term, \(\chi_i\), is an individual specific level of income, which is realized at birth, with mean one. This component captures the heterogeneity in labor income within a generation.

The third term, \(G(t - s)\), captures the life-cycle profile of earnings of households. The function \(G\) is a sum of exponentials normalized so that aggregate earnings equal \(\omega Y_t\) at each point in time, i.e.

\[
\int_{-\infty}^{t} (\delta + \eta)e^{-(\delta + \eta)(t-s)}G(t - s)ds = 1
\]

The rest of the endowment \((1 - \omega)Y_t\) is paid by claims to the representative firm.

\(^{17}\)One cannot solve the model using the optimal planner problem because of the OLG setup.
Preferences. Agents have recursive preferences as defined by Duffie and Epstein (1992). They are the continuous-time versions of the recursive preferences of Epstein and Zin (1989). For an agent $i$ with a consumption process $C_i = \{C_{it} : t \geq 0\}$, their utility $U_i = \{U_{it} : t \geq 0\}$ is defined recursively by:

$$U_{it} = E_t \int_t^{+\infty} f_i(C_{is}, U_{is}) ds$$

(6)

$$f_i(C, U) = \frac{1}{1 - \frac{1}{\psi}} \left( \frac{C^{1 - \frac{1}{\psi}}}{((1 - \gamma)U)^{\frac{\gamma - \delta}{\gamma}}} - (1 - \gamma)(\rho + \delta)U \right)$$

(7)

These preferences are characterized by three parameters: the subjective discount rate $\rho$, the coefficient of relative risk aversion (RRA) $\gamma$, and the elasticity of intertemporal substitution (EIS) $\psi$.

There are two types of agents, labeled A and B, that can differ with respect to their relative risk aversion $\gamma_j$, and their elasticity of intertemporal substitution is $\psi_j$. I denote A the risk-tolerant agent, i.e. $\gamma_A < \gamma_B$. In calibrations explored below, this agent will also be more willing to substitute consumption over time, i.e. $\psi_A \geq \psi_B$. Finally, at every point in time a proportion $\pi_A$ of newly born agents are of type A.

Markets. Households can trade two assets: claims to the representative firm and instantaneous risk-free claims (in zero net supply). The price of both of those claims is determined in equilibrium. Because markets are dynamically complete, there is a unique stochastic discount factor $\Lambda_t$:

$$\frac{d\Lambda_t}{\Lambda_t} = -r_t dt - \kappa_t dZ_t$$

where $r_t$ is the risk free interest rate and $\kappa_t$ is the price of aggregate risk. The cumulative return of the representative firm can be written:

$$\frac{dR_t}{R_t} = (r_t + \kappa_t \sigma_R) dt + \sigma_R dZ_t$$

Household Problem. Denote $A_{it}$ the financial wealth of agent $i$ at time $t$. As in Blanchard (1985), agents can access a market for annuities. There are life insurance companies that collect the agents’ financial wealth when they die. In exchange, agents receive an income stream equal to $\delta A_{it}$ per unit of time.

The problem of households is as follows. An household $i$ born at time $s(i)$ chooses a consumption path $C_i = \{C_{it} : t \geq s(i)\}$ and an amount of dollars invested in the representative firm $\theta_i = \{\theta_{it} : t \geq s(i)\}$.
to maximize his lifetime utility

\[ V_{it} = \max_{C_i, \theta_i} U_{it}(C_i) \]

subject to the dynamic budget constraint

\[ A_{is(i)} = 0 \]  \hspace{1cm} \text{(8)}

\[ dA_{it} = (L_{it} - C_{it} + (r_t + \delta)A_{it} + \theta_{it}(\mu_{Rt} - r_t))dt + \theta_{it}\sigma_{Rt}dZ_t \text{ for all } t \geq s(i) \]  \hspace{1cm} \text{(9)}

**Human Capital.** I now make a useful change of variable. Denote \( H_{it} \) the human capital of household \( i \), i.e. the present value at time \( t \) of the labor income of agent \( i \):

\[ H_{it} = E_t \left[ \int_t^{+\infty} \frac{\Lambda^u}{\Lambda^t} e^{-\delta(u-t)} L_{iu} du \right] \]  \hspace{1cm} \text{(10)}

Define \( W_{it} \), the total wealth of household \( i \) as the sum of his financial wealth \( A_{it} \) and his human capital \( H_{it} \). The wealth at birth, \( W_{is(i)} \), is given by \( W_{is(i)} = \chi_i \phi_{s(i)} Y_{s(i)} \) where \( \phi_t \) is defined as

\[ \phi_t = E_t \left[ \int_t^{+\infty} \frac{\Lambda^u}{\Lambda^t} e^{-\delta(u-t)} \omega Y_u^u Y_t G(u-t) \right] \]  \hspace{1cm} \text{(11)}

The household problem can now be reformulated as follows. Household \( i \) chooses a consumption rate \( c_{it} = \{c_{it} = C_{it}/W_{it} : t \geq s(i)\} \) and a wealth exposure to aggregate shocks \( \sigma_{W_{it}} = \{\sigma_{W_{it}} : t \geq s(i)\} \) such that for all \( t \geq s(i) \)

\[ V_{it} = \max_{c_{it}, \sigma_{it}} U_{it}(c_{it}W_{it}) \]  \hspace{1cm} \text{(12)}

\[ \text{s.t.} \quad \frac{dW_{it}}{W_{it}} = \mu_{W_{it}} dt + \sigma_{W_{it}} dZ_t \]

\[ \text{with } \mu_{W_{it}} = r_t + \delta + \kappa_t \sigma_{W_{it}} - c_{it} \]  \hspace{1cm} \text{(13)}

**3.2 Law of Motion of Household Wealth**

I first characterize the law of motion of households’ wealth. This is an important object to study because it determines the dynamics of the wealth distribution.

Households with the same preference parameters face the same trade-off, irrespective of their wealth or age, due to the homogeneity of the utility function and the constant death rate. In particular, the consumption rate \( c_{it} \) and the wealth exposure \( \sigma_{W_{it}} \) are the same for all agents in the same group \( j \in \{A, B\} \).
Denote $p_{jt}$ the wealth-to-consumption ratio of an agent in group $j$. Conjecture that the process $p_{jt}$ follows a diffusion process:

$$\frac{dp_{jt}}{p_{jt}} = \mu_{p_{jt}} dt + \sigma_{p_{jt}} dZ_t$$

(14)

**Proposition 1** (Law of Motion for Households Wealth). The wealth of households in group $j \in \{A, B\}$ follows the law of motion

$$dW_{jt} = \mu_{W_{jt}} dt + \sigma_{W_{jt}} dZ_t$$

(15)

where $\mu_{W_{jt}}$ and $\sigma_{W_{jt}}$ are given by

$$\sigma_{W_{jt}} = \frac{\kappa_t}{\gamma_j} + \frac{1}{\psi_j - 1} \sigma_{p_{jt}} + H_{jt}$$

(16)

$$\mu_{W_{jt}} = \psi_j (r_t - \rho) + \frac{1 + \psi_j}{\sqrt{2}} \kappa_t^2 + \frac{\psi_j - \psi_j}{\psi_j - 1} \kappa_t \sigma_{p_{jt}} + \frac{1}{\sqrt{2} \psi_j - 1} \sigma_{p_{jt}}^2 + \frac{1}{\psi_j - 1} \sigma_{p_{jt}}$$

(17)

The geometric volatility of wealth $\sigma_{W_{jt}}$ is the sum of two terms. The first term, the myopic demand, equals the ratio of the market price of risk to the relative risk aversion $\gamma_j$. The lower the relative risk aversion $\gamma_j$, the higher the myopic demand. The second term, the intertemporal hedging demand $H_{jt}$, captures deviations from the mean-variance portfolio due to changes in investment opportunities. If expected returns are countercyclical, this term is positive as long as $\gamma_j > 1$.

The geometric of wealth $\mu_{W_{jt}}$ is the sum of three terms. The first term is a standard term due to intertemporal substitution, determined by the EIS $\psi_j$ and the difference between the interest rate $r_t$ and the subjective discount rate $\rho$: $\psi_j (r_t - \rho)$. Note that it is similar to the term for an infinite horizon investor: the OLG setup does not change the law of motion of individual wealth in response to a given set of prices. While agents in an OLG economy have an increased discount rate $\rho + \delta$ due to the probability of death, they also face an increased effective interest rate due to annuities, $r + \delta$.

The second term corresponds to the effect of higher risk exposure on wealth growth. Agents with lower risk aversion invest disproportionately in risky assets. Due to the compensation for holding more risk, they earn, on average, higher returns. This affects their consumption rate, through a combination of an income and a substitution effect. As $\psi_j$ rises, the substitution effect becomes increasingly important, which magnifies the effect of risk aversion on total wealth growth. The third term $\Phi_{jt}$ captures changes in investment opportunities.
3.3 Prices

**Market Clearing** The market clearing for consumption is

\[
\int_{i \in I_{A_t}} C_{it}d\bar{i} + \int_{i \in I_{B_t}} C_{it}d\bar{i} = Y_tN_t
\]  

(18)

Denoting \(p_t\) the ratio of total wealth to total consumption, i.e.

\[
p_t = \frac{\int_{i \in I_{A_t}} W_{it}d\bar{i} + \int_{i \in I_{B_t}} W_{it}d\bar{i}}{\int_{i \in I_{A_t}} C_{it}d\bar{i} + \int_{i \in I_{B_t}} C_{it}d\bar{i}}
\]  

(19)

Market clearing for consumption can be rewritten as

\[
\int_{i \in I_{A_t}} W_{it}d\bar{i} + \int_{i \in I_{B_t}} W_{it}d\bar{i} = p_tY_tN_t
\]  

(20)

**Markov Equilibrium** For the purpose of determining prices, we can abstract from the distribution of wealth within each group: we only need to keep track of the share of aggregate wealth that belongs to the agent in group A:

\[
x_t = \frac{\int_{i \in I_{A_t}} W_{it}d\bar{i}}{\int_{i \in I_{A_t}} W_{it}d\bar{i} + \int_{i \in I_{B_t}} W_{it}d\bar{i}}
\]  

(21)

Because agents in group A choose a different wealth exposure compared to agents in group B, the process \(x_t\), is stochastic. The next proposition characterizes the law of motion of \(x_t\).

**Proposition 2.** The law of motion of \(x\) is

\[
dx_t = \mu_{xt}dt + \sigma_{xt}dZ_t
\]  

(22)

where \(\mu_{xt}\) and \(\sigma_{xt}\) are given by

\[
\sigma_{xt} = x_t(\sigma_{W_{At}} - \sigma - \sigma_{p_t})
\]  

(23)

\[
\mu_{xt} = x_t(\mu_{W_{At}} - \mu - \mu_{p_t} - \sigma \sigma_{p_t}) - \sigma_{xt}(\sigma + \sigma_{p_t}) + (\delta + n)(\pi_A \phi_t - x_t)
\]  

(24)

The volatility of \(x\) is given by the difference between the wealth volatility of agents in group A and the volatility of aggregate wealth. The drift of \(x\) is the sum of three terms. The first term is the difference between the wealth drift of agents in group A and the wealth drift of the economy. The second term is an Ito correction term. The third term is due to the OLG setup. It is due to the difference between the average wealth of newborns \(\phi_t\) and the average wealth of households in group A that die, \(x_t/\pi_A\). This OLG term ensures that \(x_t\) is stationary: because the volatility of \(x\), \(\sigma_x\), is zero at the boundaries 0 and 1, and its drift \(\mu_x\) is positive at 0 and negative at 1, no group of agents dominates the economy in the long run.
Market Price of Risk  The second step of our basic mechanism is that, when more wealth falls into the hands of risk-tolerant households, stock prices increase and future returns are lower. To gain some intuition on this relationship in the model, I now consider the determination of the equilibrium price of risk $\kappa_t$.

Applying Ito’s lemma on Equation (20), one obtains that the wealth-weighted average of individual wealth volatility equals the total quantity of risk:

$$x_t \sigma_{W_{At}} + (1 - x_t) \sigma_{W_{Bt}} = \sigma + \sigma_{p_t}$$  \hspace{1cm} (25)

Substituting the volatility of individual wealth from Equation (16), one obtains the market price of risk in terms of individual RRA:

$$\kappa_t = \Gamma_t(\sigma + \sigma_{p_t}) - H_t$$  \hspace{1cm} (26)

where $\Gamma_t$ corresponds to the aggregate RRA and $H_t$ corresponds to the aggregate hedging demand:

$$\frac{1}{\Gamma_t} = \frac{x_t}{\gamma_A} + \frac{1 - x_t}{\gamma_B}$$  \hspace{1cm} (27)

$$H_t = x_t H_{At} + (1 - x_t)H_{Bt}$$  \hspace{1cm} (28)

The market price of risk $\kappa_t$ is the product of the aggregate RRA $\Gamma_t$ times the total quantity of risk $\sigma + \sigma_{p_t}$, minus the total demand for risk due to the hedging.

The aggregate RRA $\Gamma_t$ is a wealth-weighted harmonic mean of individual RRAs. The higher the share of wealth owned by the agents in group A, $x_t$, the lower the aggregate risk aversion $\Gamma_t$. Therefore, ignoring for a moment the hedging demand, an increase in the fraction hold by $x_t$ decreases the market price of risk $\kappa_t$.

Risk-Free Rate  Applying Ito’s lemma on Equation (20), one obtains that the wealth-weighted average of individual wealth growth, plus a OLG term, equals the endowment growth:

$$x_t \mu_{W_{At}} + (1 - x_t) \mu_{W_{Bt}} + (\delta + n)(\phi_t - 1) = \mu + \mu_{p_t} + \sigma \sigma_{p_t}$$  \hspace{1cm} (29)

The OLG term depends on the death rate and population growth times the difference between the relative wealth of newborn households $\phi$ and one. Intuitively, because the wealth of newborns does not equal the wealth of deceased households, there is a wedge between the average individual wealth growth and the aggregate growth.
Substituting the drift of individual wealth from Equation (17), one obtains the risk-free rate in terms of individual EIS:
\[
    r_t = \rho + \frac{1}{\Psi_t} \left( \mu + \mu_{pt} + \sigma \sigma_{pt} - (\delta + n) (\phi_t - 1) - \left( \frac{1 + \psi_A}{2\gamma_A} + \left( 1 - x \right) \frac{1 + \psi_B}{2\gamma_B} \right) \kappa_t^2 - \Phi_t \right) \tag{30}
\]
where \(\Psi_t\) corresponds to the aggregate EIS and \(\Phi_t\) corresponds to aggregate changes in investment opportunities:
\[
\Psi_t = x_t \psi_A + (1 - x_t) \psi_B \tag{31}
\]
\[
\Phi_t = x_t \Phi_A + (1 - x_t) \Phi_B \tag{32}
\]
The higher the share of wealth owned by the agents in group A, \(x_t\), the closer the aggregate elasticity of substitution to \(\psi_A\). If agents in group A are also more willing to substitute inter-temporally, the risk-free rate tends to decrease as their wealth share \(x_t\) increases.

**Volatility of Asset Prices** One reason we are interested in models with heterogeneous agents is that they can potentially explain the excess volatility of asset prices. As the share of wealth owned by agents in group A, \(x_t\), fluctuates, both the market price of risk and the interest rate fluctuate, which generates fluctuations in the price-dividend ratio.

Consider \(R_t\) the cumulative return of the representative firm, and denote \(p_{Dt}\) the price-dividend ratio in this economy. One can write the cumulative return of the representative firm as:
\[
\frac{dR_t}{R_t} = (r_t + \kappa_t \sigma_{Rt}) dt + (\sigma + \sigma_{p_{Dt}}) dZ_t \tag{33}
\]
The excess volatility of the return is given by \(\sigma_{p_{Dt}}\), the geometric volatility of the price-dividend ratio. I propose a continuous-time version of the Campbell and Shiller (1988) decomposition for the volatility of the price-dividend ratio:

**Proposition 3** (Campbell-Shiller in Continuous-Time Model). The volatility of the price-dividend ratio, \(\sigma_{p_{Dt}}\), is given by:
\[
\sigma_{p_{Dt}} \approx -\int_0^\infty e^{-\alpha t} \frac{\partial E[r_t|x_0=x]}{\partial x} \sigma_x dt - \int_0^\infty e^{-\alpha t} \frac{\partial E[\kappa_t \sigma_{Rt} - \frac{1}{2} \sigma_{Rt}^2|x_0=x]}{\partial x} \sigma_x \tag{34}
\]
with \(\alpha = \exp(-E[\log p_t^D])\).
The volatility of the price-dividend ratio can be decomposed into two terms: the first term depends on the volatility of expectations of future risk-free rates. The second term depends on the volatility of expectations of future expected excess log-returns.

By exchanging the expectation and the derivative operations in the RHS of Equation (34), one obtains:

$$
\sigma_{pDt} \approx - \int_0^{+\infty} e^{-\alpha t} E \left[ \frac{\partial x_t}{\partial x_0} \bigg| x_0 = x \right] \sigma_x - \int_0^{+\infty} e^{-\alpha t} E \left[ \frac{\partial x_t}{\partial x_0} \left( \kappa \sigma_R t - \frac{1}{2} \sigma^2_R t \right) \bigg| x_0 = x \right] \sigma_x
$$

where $\partial x_t/\partial x_0$, denotes the first-variation of the process $x_t$. This equation expresses the risk-free rate channel directly in terms of the derivative of the risk-free rate around $x$, and the excess returns channel in terms of the derivative of expected excess log-returns around $x$. This suggests that, the higher the derivative of the risk-free rate and of the market price of risk with respect to the group share $x$, the larger the volatility of asset prices.

In turn, as shown in (26) and (30), the derivatives of the risk-free rate and of the market price of risk depend critically on the derivative of the aggregate RRA $\Gamma$ and of the inverse of the aggregate EIS $1/\Psi$ with respect to the group share $x$:

$$
\frac{\partial \Gamma}{\partial x} = \Gamma^2 \left( \frac{1}{\gamma_B} - \frac{1}{\gamma_A} \right) \quad (37)
$$

$$
\frac{\partial 1/\Psi}{\partial x} = \Psi^{-2} (\psi_B - \psi_A) \quad (38)
$$

Together, these equations relate the volatility of asset prices to the degree of preference heterogeneity across households. Importantly, these derivatives themselves decrease in $x$ (in absolute value); this suggests the volatility of asset prices in the model is higher for low value of the group share $x$.

An important question is whether the degree of households heterogeneity necessary to generate volatile asset prices in the model is consistent with the data. The next section examines how moments about the wealth distribution help discipline household heterogeneity.

---

18Formally, the first-variation process $D_t = \partial x_t/\partial x_0$ associated to the diffusion $x_t$ satisfies:

$$
D_0 = 1 \quad (35)
$$

$$
\frac{dD_t}{Dt} = \frac{\partial \mu_x}{\partial x}(x_t) dt + \frac{\partial \sigma_x}{\partial x}(x_t) dZ_t \quad (36)
$$

See for instance Fournié et al. (1999).
4 Wealth Distribution

In this section, I examine the wealth distribution implied by the model, which is a key contribution of this paper. The key difference with the existing literature on wealth inequality is that, in the model, the dynamics of individual wealth depends on the aggregate state of the economy. Therefore, wealth inequality moves over time, as in the data. Moreover, the distribution exhibits a fat tail, driven by the high average returns of top households.

4.1 Dynamics of Wealth Density

Denote $w_{it}$ the wealth of agent $i$ relative to the per-capita wealth in the economy, i.e. $w_{it} = W_{it}/(p_tY_t)$. Applying Ito’s lemma, the law of motion of the relative wealth $w_{it}$ is

$$\frac{dw_{it}}{w_{it}} = \mu_{w_{it}}dt + \sigma_{w_{it}}dZ_t$$

(39)

where $\mu_{w_{it}}$ and $\sigma_{w_{it}}$ are given by

$$\sigma_{w_{it}} = \sigma_{W_{it}} - \sigma - \sigma_{p_t}$$

(40)

$$\mu_{w_{it}} = \mu_{W_{it}} - \mu - \mu_{p_t} - \sigma_{p_t} - \sigma_{w_{it}}(\sigma + \sigma_{p_t})$$

(41)

I first characterize the dynamics of the wealth density in the model. Denote $g_{jt}$ the density of relative wealth within each group of agent $j \in \{A,B\}$, and $g_t$ the density of relative wealth across all households, i.e.

$$g_t = \pi_A g_{At} + (1 - \pi_A) g_{Bt}$$

(42)

Finally, denote $g_{\chi t}$ the distribution of relative wealth for newborn agents.\(^{19}\) The next proposition characterizes the law of motion of the wealth density within each group $g_{jt}$ for $j \in \{A,B\}$:

**Proposition 4** (Kolmogorov-Forward Equation with Aggregate Shocks). **The law of motion of** $g_{jt}$ **is given by**

$$dg_{jt} = \left( -\mu_{w_{jt}}\partial_w(wg_{jt}) + \frac{1}{2} \sigma_{w_{jt}}^2 \partial_{ww}(w^2g_{jt}) + (\delta + n)(g_{\chi t} - g_{jt}) \right) dt - \sigma_{w_{jt}}\partial_w(wg_{jt})dZ_t$$

(44)

\(^{19}\) Denoting $g_{\chi}$ the density of $\chi$, defined in Section 3, we have

$$g_{\chi t}(w) = \frac{1}{\phi_t} g_{\chi} \left( \frac{1}{\phi_t} w \right).$$

(43)
Given the evolution of individual wealth \((\mu_{wjt} dt, \sigma_{wjt} dZ_t)\), this equation gives the evolution of the wealth density \(g_{jt+dt} - g_{jt}\). The key difference from the existing literature is that, because households choose different exposures to aggregate shocks (i.e. \(\sigma_{wjt} \neq 0\)), the wealth density is stochastic.

4.2 Tail Index

While a full characterization of the entire wealth distribution is not feasible, I show that one can characterize analytically its right tail.

**Tail Index** By analogy with the case of a static distribution, I define the tail index of a stochastic distribution as the smallest number \(\zeta\) for which moments of order higher than \(\zeta\) converge to infinity.

**Definition 1 (Fat-Tail).** For a density \(g_t\) and \(\xi > 0\), denote \(m_\xi^t\) the moment of order \(\xi\):

\[
m_\xi^t = \int_0^{+\infty} w^\xi g_t(w)dw
\]

A density \(g_t\) is fat-tailed with tail index \(\zeta\) if there exists \(0 < \zeta < +\infty\) such that moments of order higher than \(\zeta\) converges to infinity almost surely, i.e.

\[
\zeta = \inf \left\{ \xi \in \mathbb{R}_+ \text{ s.t. } \lim_{t \to +\infty} m_\xi^t = +\infty \text{ a.s.} \right\}
\]

The next proposition characterizes the tail index of the wealth distribution in the economy.

**Proposition 5 (Tail Index).** Assume that \(x_t\) is irreducible positive recurrent with unique invariant probability measure. Denote

\[
\zeta = \frac{\delta + n}{E[\mu_{wA_t} - \frac{1}{2} \sigma_{wA_t}^2]}
\]

where \(E\) denotes the expectation with respect to the invariant probability measure of \(x\). If the following conditions are satisfied:

1. Agents in group A grow on average faster than the economy and that agents in group B, i.e.

\[
E \left[ \mu_{wA_t} - \frac{1}{2} \sigma_{wA_t}^2 \right] > \max \left( 0, E \left[ \mu_{wB_t} - \frac{1}{2} \sigma_{wB_t}^2 \right] \right)
\]

2. The distribution of human capital \(\chi_i\) has a tail thinner than \(\zeta\).
Then the wealth distribution is fat-tailed with tail index $\zeta$. The distribution of financial wealth is also fat-tailed with tail index $\zeta$.\textsuperscript{20}

Moreover, the wealth distribution of agents in group A is fat-tailed with tail index $\zeta$, whereas the wealth distribution of agents in group B has a thinner tail.

The tail index of the wealth distribution $\zeta$ can be expressed as the ratio of two terms. The numerator is the sum of the death rate and of the population growth rate, $\delta + n$. The denominator is the average logarithmic wealth growth rate of agents in group A relative to the rest of the economy,

$$E[\ln w_{At}] = \mu w_{At} - \frac{1}{2}\sigma^2 w_{At}.$$ \textsuperscript{21}

This formula extends existing results in the literature on two dimensions.\textsuperscript{21} First, it gives the tail index of the distribution in an economy where the dynamics of individual wealth varies over time (this is because, in this economy, the interest rate and the market price of risk vary over time). In this setup, the formula shows that the tail index depends on the average wealth growth of top households. Second, it gives the tail index of the distribution in an economy where households have heterogeneous exposure to aggregate shocks. In this setup, the formula shows that the tail index depends on the average growth of the logarithmic wealth $E[\mu w_{At} - \frac{1}{2}\sigma^2 w_{At}]$. While the exposure of top households to aggregate risk increases their average average geometric growth $E[\mu w_{At}]$, this has a dampened effect on the tail index due to the associated excess volatility $-\frac{1}{2}E[\sigma^2 w_{At}]$.

It is enlightening to compare this result to the case of infinite-horizon economies. In infinite-horizon economies, a stationary wealth distribution obtains only when the logarithmic relative wealth growth rate of agents in group A is zero on average.\textsuperscript{22} OLG models break this equivalence, by considering death and population growth. What Proposition 5 shows is that the growth rate of agents in group A is still a key statistic: it controls how thick the right tail of the wealth distribution is.

Proposition 5 shows, as long the tail index of the labor income distribution is higher than the tail index of wealth distribution,\textsuperscript{23} then the distribution of financial assets has the same tail index as the distribution of wealth. Moreover, this tail index does not depend on the level of labor inequality.

\textsuperscript{20}The financial wealth of an household is defined as his total wealth minus the capitalized value of his future labor income, following Section 3.

\textsuperscript{21}Wold and Whittle (1957) is the first paper studying the tail index of the wealth distribution.

\textsuperscript{22}See for instance Blume et al. (1992).

\textsuperscript{23}This is the case empirically: the wealth distribution has a tail index of 1.5 while the distribution of labor income has a tail index between 2 and 3. See for instance Toda (2012).
This is a classic result for deterministic economies. The intuition is that, for households in the right tail of the distribution, most of the wealth is held in the form of financial wealth, rather than human capital. In short, the tail index is a moment interesting to match, because it is does not require to take a strong stance on how to measure human capital, or its distribution across households.

**Law of motion of $x_t$** As shown in Section 3, the growth of $x_t$ is the sum of the relative wealth growth of agents in group $A$ and an OLG term, that depends on $x_t$ as well as the relative wealth of a newborn agent $\phi_t$. Because $x_t$ is ergodic, this gives a relationship between the relative wealth growth of agent in group $A$ and the average level of $x$.

**Proposition 6 (Relation Between Tail Index and Group Share).** Under the assumptions of Proposition 5, the tail index $\zeta$ of the wealth distribution is given by:

$$\zeta = \frac{1}{1 - \mathbb{E} \left[ \frac{\pi_A}{x_t} \phi_t \right]}$$

where $\mathbb{E}$ denotes the expectation with respect to the stationary density of $x$. In particular $\zeta \geq 1$, i.e. the tail of the distribution is less thick than Zipf’s law.

The formula shows that the tail index $\zeta$ is directly related to $\mathbb{E} \left[ \frac{\pi_A}{x_t} \phi_t \right]$, i.e. the average ratio of the relative wealth of a newborn agent ($\phi_t$) to the average wealth of an agent in group $A$ ($x_t/\pi_A$). This means that, in equilibrium, the tail index $\zeta$, which is measure of inequality within agents in group $A$, is directly related to $x_t/\pi_A$, which is a measure of inequality between the average household in group $A$ and the average household in group $B$.

Small changes in the tail index correspond to large changes in this ratio. For instance, a tail index close to 1 (i.e. Zipf’s law) corresponds to an economy where the average wealth of an agent in group $A$ is infinitely large compared to the average human capital of a newborn. In contrast, a tail index closer to 1.5 corresponds to an economy where the average wealth of an agent in group $A$ is only three times higher than the average human capital of a newborn. Small changes in the tail index correspond to large differences in the distribution of wealth.

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24 See, for instance, Achdou et al. (2016) or Gabaix et al. (2016).

25 See also Appendix D.1.
4.3 Dynamics of Top Wealth Shares

I now relate the dynamics of the wealth in a top percentile to the dynamics of individual wealth. Let $\alpha$ be a top percentile, e.g. top 1%. Denote $q_t$ the $\alpha-$quantile, i.e.,

$$\alpha = \int_{q_t}^{+\infty} g_t(w)dw$$  \hspace{1cm} (49)

$q_t$ corresponds to the wealth of an agent exactly at the $\alpha$ percentile threshold of the distribution.

We can now define the wealth share owned by the top percentile $\alpha$, $S_t$:

$$S_t = \int_{q_t}^{+\infty} w g_t(w)dw$$  \hspace{1cm} (50)

The following proposition characterizes the dynamics of the wealth share of the top percentile $\alpha$:

**Proposition 7** (Law of Motion of Top Wealth Shares). *The law of motion of the top wealth share $S_t$ is*

$$\frac{dS_t}{S_t} = \mu_{St} dt + \sigma_{St} dZ_t$$  \hspace{1cm} (51)

where $\mu_{St}$ and $\sigma_{St}$ are given by\(^\text{26}\)

$$\sigma_{St} = E^{g_t w}[\sigma_{w_{it}} | w_{it} \geq q_t]$$  \hspace{1cm} (52)

$$\mu_{St} = E^{g_t w}[\mu_{w_{it}} | w_{it} \geq q_t] + \frac{q_t^2 g_t(q_t)}{2 S_t} \text{Var}^{g_t} [\sigma_{w_{it}} | w_{it} = q_t]$$

$$+ \left( \delta + n \right) \left( \frac{\alpha q_t}{S_t} - 1 \right) + \frac{\delta + n}{S_t} \int_{q_t}^{\infty} (w - q_t) g_{\chi t}(w)dw$$  \hspace{1cm} (53)

where the expectation $E^{g_t w}$ refer to the wealth-weighted, cross-sectional average with respect to the wealth density $g_t$.

The geometric volatility of the top wealth share, $\sigma_{St}$, is the wealth-weighted average geometric volatility of individuals in the top percentile. As $\alpha$ denotes an increasingly high percentile, there are relatively more agents of type $A$ in the top percentile, and therefore the exposure of the wealth share of top percentiles converges to $\sigma_w^A$. The model therefore generates the empirical evidence found in Section 2 (Table 1).

The geometric drift of the top wealth share, $\mu_{St}$, is the sum of three terms. The first term corresponds to the average, wealth-weighted, geometric drift of individuals at the top. The second

\(^{26}\) $E^{g_t w}$ denotes the wealth-weighted average and $\text{Var}^{g_t}$ denotes the variance with respect to the wealth density $g_t$. 

23
term is due to the heterogeneous exposure of households at the threshold. It depends on the variance of risk exposures across households at the quantile $q_t$. When a negative shock hits the economy, top wealth shares decrease a bit less than the wealth of households inside the top percentile, because some households from group $B$ enter the top. Conversely, when a positive shock hits the economy, top wealth shares increase a bit more than the wealth of households inside the top percentile, because some households from group $A$ enter the top. Therefore, heterogeneous exposure to aggregate shocks tends to increase the growth of top wealth shares due to a composition effect. The third term is due to the death of individuals at the top, as well as the eventual birth of households with human capital higher than $q_t$.

In particular, while changes in the composition of households in the top affect the average growth of top wealth shares, they do not change the exposure of top wealth shares to stock market returns. In other words, the exposure of top wealth shares to aggregate shocks, as measured in Section 2, recovers the average exposure of individual households inside the top percentile.

5 Fitting the Model to the Data

I now bring the model to the data. Qualitatively, the model generates the two key facts about wealth inequality and the wealth distribution (i) top wealth shares increase when stock market returns are high (ii) top wealth shares predict future excess returns. However, I highlight a key tension to fit the model to the data: to match the high volatility of asset prices, the model requires such a large degree of heterogeneity that it generates a wealth distribution with a right tail thicker than the data.

5.1 Estimation Method

Method I estimate the parameters of the model by minimizing the distance between moments from the data and those implied by the model. I proceed as follows. I select a vector of moments $m$ computed from the actual data. Given a candidate set of parameters $\Theta$, I solve the model, and compute the moments $\hat{m}(\Theta)$. I search the set of parameters $\hat{\Theta}$ that minimizes the weighted deviation between the actual and model-implied moments $J(\Theta)$, i.e.

$$J(\Theta) = (m - \hat{m}(\Theta))'W(m - \hat{m}(\Theta))$$

$$\hat{\Theta} = \arg \min_{\Theta} J(\Theta)$$
where $W$ is a weight matrix, computed as inverse of the variances of moments as measured in the data. The moments in the models are based on 500 simulated samples with 100 years of data to match the length of the actual data.\(^{27}\)

**Asset Prices** Following Gârleanu and Panageas (2015), I use four asset price moments, corresponding to the average and standard deviation of the risk-free rate and of stock market returns (in real term). The data for the average equity premium, the volatility of returns, and the average interest rate are from Shiller (2015). The volatility of the real risk-free rate is inferred from the yields of 5-year constant maturity TIPS following Gârleanu and Panageas (2015).

Following the approach of Barro (2006), I compare the equity premium and the standard deviation of the stock market return in the data to the returns of a firm with a debt-equity ratio equal to the historically observed debt-equity ratio for the U.S. non financial corporate sector, $\lambda \approx 0.5.\(^{28}\)

**Wealth Inequality** Compared to Gârleanu and Panageas (2015), I add two moments that to discipline the wealth dynamics of the households in the right tail of the distribution.

The first moment is the tail index of the distribution of financial wealth. As shown in Proposition 5, the moment disciplines the average logarithmic wealth growth rate of households in group $A$ relative to the economy. I use a MLE method to estimate the moment in the model and in the simulated data.\(^{29}\) In the data, I measure a tail index of $\zeta = 1.5$, consistent with previous studies.\(^{30}\)

Graphically, Figure 2 plots the log percentile as a function of the log net worth for the U.S. distribution, in the SCF and in Forbes 400 data. The linear slope is characteristic of a distribution with a Pareto tail.

The second moment is the exposure of financial wealth of the top 0.01% to stock market returns, measured in Table 1. This moment captures the wealth exposure of households in group $A$ to the stock market, as indicated by Proposition 7. However, there are two caveats: whereas Proposition

\(^{27}\)For each sample, I simulate the model for 200 years, starting from a random draw from the stationary distribution of $x$, and I throw away the first 100 years. Increasing the number of years, increasing the number of samples, or diminishing the frequency do not significantly change the results.

\(^{28}\)Denoting $\sigma_{R_t}$ the geometric volatility of the representative firm, the return of the levered firm follows as

$$
\frac{d\tilde{R}_t}{\tilde{R}_t} = (r_t + \kappa_t \lambda \sigma_{R_t}) dt + \lambda \sigma_{R_t} dZ_t
$$

\(^{29}\)See Goldstein et al. (2004).

\(^{30}\)See, for instance, Klass et al. (2006) or Vermeulen (2018).
tion 7 characterizes the exposure of total wealth over a short time period, I can only measure the wealth exposure of financial wealth over a four year horizon. While the difference is negligible for households the right tail,\textsuperscript{31} to avoid any discrepancy, I measure the moment implied by the model using the same methodology as in the data, by regressing the log growth of financial wealth in the top 0.01% on excess stock market returns over a four year horizon.

**Calibrated Parameters** The law of motion of the endowment process per capita is $\mu = 2\%$ and $\sigma = 4.1\%$. The death rate is $\delta = 2\%$. The population growth rate is $n = 1\%$. Finally, human capital is calibrated based on U.S. data about life-cycle labor income, following Gârleanu and Panageas (2015).\textsuperscript{32}

**Estimated Parameters** The model has 6 remaining parameters that I estimate. 2 parameters correspond to the preference parameters specific to households in group A ($\gamma_A, \psi_A$) and 2 parameters correspond to the preference parameters specific to households in group B ($\gamma_B, \psi_B$). I impose sensible restrictions on the preference parameters of agents in group B, i.e. $\psi_B \geq 0.05$ and $1/\gamma_B \geq 0.05$. The remaining parameters is the subjective discount rate $\rho$ and the population share of the agents in group A, $\pi_A$.

### 5.2 Results

I now bring the model to the data. Qualitatively generates the stylized facts about the joint dynamics of wealth inequality and asset prices documented in Section 2. However, to generate volatile asset prices, the model requires a wealth distribution with a tail much thicker than the data.

**Estimation on Asset Prices Only** I first estimate the model on asset price moments, which is a very similar compared to to Gârleanu and Panageas (2015). Column (2) of Table 5 reports the result of the estimation. Figure 3 plots the market price of risk, the interest rate, the stationary density of the state variable, and the Campbell-Shiller decomposition of the volatility of the price-dividend ratio.\textsuperscript{33} The model matches very well asset price moments: it generates a high equity

\textsuperscript{31}This is because their human capital tends to be much smaller than their financial wealth, see Appendix D.1
\textsuperscript{32}See Appendix D.
\textsuperscript{33}The only difference is that I consider an economy where population grows at rate 1%. That being said, the result of the estimation is very similar to Gârleanu and Panageas (2015), both in terms of preference parameters and in
premium, with a high volatility, together with a low risk-free rate with low volatility.

The model can match the high volatility of asset prices because it has a high degree of preference heterogeneity, both in terms of RRA ($\gamma_A \approx 1.5$ vs $\gamma_B \approx 17$) and in terms of EIS ($\psi_A \approx 0.8$ vs $\psi_B = 0.05$). Due to this high degree of preference heterogeneity, the market price of risk and the risk-free rate are very sensitive to the wealth share of group $A$ (as explained in Section 3). Indeed, Figure 3 show that both prices decrease sharply in the wealth share of group $A$. This sensitivity, combined with the fact that the model spends a large amount of time in region where $x$ is low, i.e. where the sensitivity of these prices is high, generates a large volatility of asset prices: Figure 3 plots the excess volatility of returns, as a sum of a risk-free rate channel and an excess returns channel. Overall, 75% of excess volatility is due to the expected excess-return channel, while 25% is due to the risk-free rate channel. The possibility that agents in group $A$ may be “wiped out” after a series of negative shocks, i.e. that their wealth share $x$ may approach zero, drives a large amount of asset prices volatility, even in normal times.

I now assess whether the model fits moments about the wealth distribution, that were not targeted in the first estimation. I find that the model targets well the exposure of top wealth shares to stock market returns: it is 1.0 in the model compared to 0.95 in the data. More importantly, the model overestimates the thickness of the tail of the distribution: in the model, the tail index of the wealth distribution is 1.1, whereas it is close to 1.5 in the data. Visually, Figure 2 shows that the right tail of the wealth distribution is much thicker in the model than the data.

In the model, agents in group $A$ grow much faster than the rest of the distribution, which leads to a wealth distribution with a thick tail (Proposition 5). As seen in Proposition 6, this thick tail corresponds to a high ratio of the wealth share of agents in group $A$ relative to their population share. Indeed, in the model, while the population share of agents in group $A$ is only 1%, their wealth share hovers around 20% (i.e. agents in group $A$ are 20 times more wealthy than the average household).34

Adding Wealth Inequality Moments To ask whether the model can match asset prices without implying a counter-factual wealth distribution, I re-estimate the model in Column (4) of Table 5, terms of moments.

34 The fact that $x/\pi_A \approx 20\%$ does not mean that the wealth share of the top 1% is 20% in the model. Contrary to the tail index, the wealth share of the top 1% depends on the distribution of human capital in the population: households in group $B$ that are born with a high amount of initial human capital end up at the top of the distribution, whereas some households in group $A$ that born with little human capital end up at the bottom.
targeting jointly the four asset prices moments and the two moments about the wealth distribution. The model cannot jointly match asset prices and the wealth distribution: the J-statistic, which measures the distance between the model and the data, jumps from 0.05 to 15.

The reason the model fails to match both sets of moments is that there is a tension between matching the volatility of asset prices and the tail index of the wealth distribution. To match a thinner tail index, the model needs to decrease the average wealth growth of households in group A (either by increasing their subjective discount rate $\rho$ or by decreasing their EIS $\psi_A$). Because the wealth-weighted average growth of individual households has to sum up to aggregate wealth growth, this must be compensated either by an increase of the average wealth growth of agents in group B, or by an increase in the average group share of agents in group A. In either case, the average sensitivity of aggregate RRA $\Gamma$ and of the inverse of the aggregate EIS $1/\Psi$ to the group share $x$ must decrease. In turn, this decreases the average sensitivity of the market price of risk and of the risk-free rate to the group share, which ultimately decreases the volatility of asset prices.

Column (4) reports the results of model estimated on all moments. The model overestimates the interest rate (3.2% in the model compared to 2.8% in the data). This is because, in the model, both agents have a very low EIS. Moreover, the model underestimates the equity premium (3.6% in the model compared to 5.2% in the data). The reason the model misses the equity premium, rather than the standard deviation of returns, is that this moment is less precisely estimated, and therefore is given less weight in the J-statistic. Moreover, having such a low equity premium allows the model to lower the average wealth growth of households in group A, by decreasing their compensation for holding aggregate risk. In other words, underestimating the equity premium allows the model to match the thinner tail of the wealth distribution, without substantially changing the degree of preference heterogeneity. Finally, even though this degree of heterogeneity still generates a substantial volatility in asset prices, half of the fluctuations in asset prices are now driven by fluctuations in expected risk-free rate, rather than fluctuations in expected excess stock returns, which is counter-factual.

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35 See Equation (29).
36 Indeed, Equation (37) shows that the sensitivity of aggregate RRA and of the inverse of the aggregate EIS is high when $x$ is close to zero and when $\gamma_B$ and $1/\psi_B$ are high.
37 As seen in Proposition 3.
38 See, for instance, Campbell and Shiller (1988).
5.3 Realistic Ingredients

The environment is highly stylized and one could consider additional ingredients that would make the wealth distribution more realistic. I now show that adding realistic ingredients such as idiosyncratic shocks (Benhabib et al. (2015a)), heterogeneous subjective discount factors (Krusell and Smith (1998), Carroll et al. (2017)), or bounded lives do not help the model to solve the tension described above.

Idiosyncratic Shocks  In the baseline model, there are no idiosyncratic wealth shocks: the position of households in the wealth distribution depends on their initial capital, their group, and their age. Yet, in the data, idiosyncratic shocks play an important role in driving the right tail of the distribution.39

Would incorporating this idiosyncratic volatility help the model to match the data? In the appendix, I show that adding idiosyncratic wealth volatility to the model systematically thickens the right tail of the distribution (Lemma 3). Since the baseline model already gives a right tail that is too thick compared to the data, adding idiosyncratic wealth shocks only worsens the tension between matching asset prices and the right tail of the distribution. Formally, Column (2) Table 6 re-estimates the model with an idiosyncratic wealth volatility of 10%. For simplicity, I assume that the presence of these idiosyncratic shocks does not affect the saving decisions of households.40 As predicted, the fit worsens: the J-statistic increases to 21.

Heterogeneity in Discount Rates  While the model considers heterogeneity in RRA and EIS, households have the same subjective discount rate $\rho$. One may wonder whether considering heterogeneity in subjective discount rate $\rho$, as in Krusell and Smith (1998) or Carroll et al. (2017), could help the model better fit the data. More generally, allowing the subjective discount rate to differ across agents is a reduced form way of allowing agents to have different motives to save, due to precautionary saving, or bequests motive. To check whether this changes the results, I report the result of the estimated model in Column (3) of Table 6. I find very similar estimates compared to the model where agents have homogeneous discount rates. This is because, in the estimated

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39See, for instance, Benhabib et al. (2015a) and Gomez (2018), that stress that idiosyncratic volatility is necessary to match wealth mobility. For recent asset prices models where idiosyncratic wealth shocks play an important role, see Kogan et al. (Forthcoming) and Gârleanu and Panageas (2017).

40Precautionary saving would tend to increase the saving rate of households, which corresponds to a rise in their subjective discount rate $\rho$. 

29
model, the EIS of agents in group B $\psi_B$ is so low that the exact value of their subjective discount rate $\rho_B$ has very little impact on their saving decision (Proposition 1), and therefore on aggregate quantities.

**Bounded Lives** To get analytical results on the tail index of the wealth distribution, the model assumes that the death probability of households does not depend on age. In particular, this means that, in the model, certain households live for a large amount of time and end up with a disproportional amount of wealth compared to the rest of the economy. In reality, households usually die after a certain age, which limits how rich richer households can really be.

In a static model, where the wealth distribution is in a steady state, one can show that this assumption has no impact on the tail index of the wealth distribution.\(^{41}\) This is because the tail index is a local property, that characterizes the slope of the wealth density. Imposing households to die after a certain age only results in a truncated Pareto distribution, with the same tail index as the original model.

To check that this still holds true in a model with aggregate shocks, I rely on simulations. More precisely, I simulate the wealth distribution in the model, with the added assumption that households older than 70 years die with probability one. I find that incorporating this more realistic “upper bound” on age does not change the tail index of the wealth distribution, measured as the average slope of the wealth density with respect to wealth.\(^{42}\)

6 Resolving the Tension

The previous section shows that the baseline model cannot match the excess volatility of asset prices without implying a wealth distribution with a right tail thicker than the data. This tension comes from the combination of two facts: (a) matching the high volatility of asset prices requires a large degree of preference heterogeneity, and (b) a large degree of preference heterogeneity gives rise to a wealth distribution with a tail thicker than the data.

I now explore two parsimonious deviations from the baseline model that help resolve this tension. In Section 6.1, I assume that preferences are transitory, which weakens the link (b) between

\(^{41}\)See, for instance, Steindl (1965)

\(^{42}\)A video of the wealth distribution in an economy with fluctuating drift is available at [http://www.matthieugomez.com/wealthinequality.html](http://www.matthieugomez.com/wealthinequality.html). Even though the wealth distribution is not in a steady state, the slope of the log density with respect to log wealth remains stable over time.
preference heterogeneity and the tail index of the wealth distribution. In Section 6.2, I augment
the model with time-varying investment opportunities for rich households compared to the rest,
which weakens the link (a) between preference heterogeneity and the volatility of asset prices.

6.1 Transitory Preferences

In the baseline model, risk-tolerant households remain risk-tolerant their whole life. I now assume
that risk-tolerant households become risk-averse after some time, i.e. that households transition
from group A to group B with a certain hazard rate. This limits the effect of preference heterogeneity
on the right tail of the wealth distribution, which helps resolve the tension described earlier.

The intuition is as follows. In the baseline model, only households in group A are present in the
right tail of the wealth distribution.\footnote{As shown in Proposition 5, the tail index of the distribution of households in group B is higher than the tail index of the distribution of households in group A.} Therefore, the right tail of the distribution directly reflects
the wealth dynamics of households in group A. In contrast, in the augmented model, both types
of households are present in the right tail of the wealth distribution. Therefore, the right tail of
the distribution reflects the dynamics of both types of households. In particular, a high average
growth rate of households in group A no longer implies a wealth distribution with a tail thicker
than the data. Similarly, a high wealth exposure of households in group A no longer implies top
wealth shares that react too much to stock market returns compared to the data.

**Reduced-Form Evidence** One key implication of this departure is that we should observe some
heterogeneity in stock-market exposure at the top of the wealth distribution. In particular, the
model predicts that younger households in top percentiles tend to be more exposed to stock market
returns compared to older households.

I test this prediction using individual data from Forbes 400. Every year, I construct the average
return of households in Forbes 400 with an average return of households with an age below the
median ($g = 1$), and the average return of households with an age above the median ($g = 2$). I then
test whether the stock market exposure of older households differs from the stock market exposure
of younger households, by estimating the following regression:

$$\sum_{t=1}^{T} \log R_{gt} - 4 \log R_{ft} = \alpha + \delta 1_{\text{Age} \geq \text{Median}} + (\beta + \gamma 1_{\text{Age}})(\log R_{Mt} - \log R_{ft}) + \epsilon_{gt}$$
where $R_{gt}$ the return of households in group $g$ at year $t$. This specification follows the aggregate specification from Table 1, but I now allow older households to have a different exposure to stock market returns, as measured by $\gamma$.

Table 7 reports the estimates for $\gamma$, in a specification without year fixed effects (Column (1)), and a specification with year fixed effects (Column (2)). I find that older households tend to be less exposed to stock market returns compared to younger households, i.e. I find $\gamma \approx -0.31$. This is consistent with the fact that risk tolerant households transition to risk averse after a certain amount of time.

**Augmented Model**  Formally, I modify the baseline model by introducing a transition rate from group $A$ to group $B$ with hazard rate $\tau$: during a short period of time $dt$, a proportion $\tau dt$ of agents switches from group $A$ to group $B$.\footnote{This trick is often used to make the distribution of wealth stationary in models of financial accelerator, such as Brunnermeier and Sannikov (2014), He and Krishnamurthy (2013), Di Tella (2016).} For parsimonious reason, I assume that agents in group $A$ do not forecast this change of preferences, so that their optimization problem remains the same.\footnote{It is not obvious whether agents should take into account the transitory nature of preferences in their optimization problem. A model where agents take into account possible change in preferences in their optimization problem would substantially complicate the model, since the utility of an agent in group $A$ would not be homogeneous in wealth anymore.}

Compared to the baseline model, the dynamics of the state variable $x_t$, the share of wealth owned by agents in group $A$, must be adjusted to account for the transition from $A$ to $B$. (24) becomes:

$$
\mu_{xt} = x_t (\mu_{w_A t} - \mu - \mu_{p_t} - \sigma_{p_t}) - \sigma_{xt}(\sigma + \sigma_{p_t}) + (\delta + n)(\pi_A \phi_t - x_t) - \tau x_t_{\text{transition}}
$$

More importantly, the transition rate also changes the tail index of the wealth distribution:

**Proposition 8** (Tail Index with Transition). Assume that $x_t$ is irreducible positive recurrent with unique invariant probability measure, and $\tau > 0$. Denote

$$
\zeta = \frac{\delta + n + \tau}{E[\mu_{w_A t} - \frac{1}{2}\sigma_{w_A t}^2]}
$$

where $E$ denotes the expectation with respect to the invariant probability measure of $x$. If the following conditions are satisfied:
1. Agents in group $A$ grow on average faster than the economy and that agents in group $B$, even after adjusting for transition i.e.

$$\mathbb{E}\left[\mu_{w_{At}} - \frac{1}{2}\sigma_{w_{At}}^2\right] > \max\left(0, \left(1 + \frac{\tau}{\delta + n}\right)\mathbb{E}\left[\mu_{w_{Bt}} - \frac{1}{2}\sigma_{w_{Bt}}^2\right]\right)$$  (58)

2. The distribution of human capital $\chi_i$ has a tail thinner than $\zeta$.

Then the wealth distribution is fat-tailed with tail index $\zeta$. The distribution of financial wealth is also fat-tailed with tail index $\zeta$. Moreover, both the wealth distribution of agents in group $A$ and in group $B$ are fat-tailed with tail index $\zeta$.

The higher the transition rate, the thinner the right tail of the distribution. Moreover, as discussed above the distribution in group $B$ “inherits” the tail index of the distribution of group $A$, i.e. both types of agents are present at the top of the distribution.

The analytical expression for the tail index in presence of transition between types was first derived by Luttmer (2011), and more recently by Gabaix et al. (2016) and Jones and Kim (2016). This proposition generalizes it to the case of an economy with a Markovian structure. As in these papers, I find that transition rates between types allows certain agents to grow very fast, without simultaneously thickening the right tail of the distribution.46

**Results** I now estimate the model augmented with a transition rate $\tau = 2\%$. This choice means that the average life of a risk tolerant agent is roughly 25 years, after which the household dies or becomes risk averse. I take this value for the sake of example — similar results obtains with higher transition rates.

I estimate the augmented model on the six moments described earlier. The first column of Table 8 shows that the augmented model can jointly match asset prices and the wealth distribution. In particular, the model can generate a high volatility of returns $\sigma_{R} = 18\%$ together with a high tail index $\zeta = 1.5$. The $J$-statistic is equal to 0.35, which is 50 times smaller than the $J$-statistic of the baseline model.

The estimated model has a higher degree of preference heterogeneity compared to the baseline model estimated in Column (1) of Table 5: the RRA of households in group $A$ equals $1.2$ (v.s. $1.5$

46 In Luttmer (2011), transition from a high growth regime to a low growth regime allows the age of big firms to be reasonably low, without making the firm size distribution thicker than Zipf’s law. In Gabaix et al. (2016) and Jones and Kim (2016), transition from a high growth regime to a low growth regime allows the transition speed between two steady states to be reasonably high, even though the distribution of income has a tail index below 3.
in the baseline model), while their EIS equals 1.0 (vs 0.7 in the baseline model). Because the state variable in the model is less persistent compared to the baseline model (see (56)), a higher degree of heterogeneity is required to match the excess volatility of asset prices (Proposition 3). Even with this higher degree of preference heterogeneity, the distribution has a thinner tail compared to the baseline model, due to the transition from group A to group B.

6.2 Time-Varying Investment Opportunities

In the baseline model, top wealth shares only move with excess stock market returns. I now augment the model with time-varying investment opportunities for the rich relative to the poor, i.e. with fluctuations in the financial returns available to the agents in group A and the financial returns available to the agents in group B. These shocks create additional, low-frequency fluctuations in wealth inequality, which increases the volatility of asset prices, without increasing the level of inequality in the long run. This helps resolve the tension present in the baseline model.

Reduced-Form Evidence To help motivate this departure from the baseline model, I compare the actual time series of top wealth shares with the one predicted from past excess stock market returns. That is, I examine the actual time series of top wealth shares with a synthetic version, constructed from a cumulative sum of past returns:

\[
\log (\text{Share Top } 0.01\%_{t+1}) = \alpha + \rho \times \log (\text{Share Top } 0.01\%_{t}) + \beta \times (\log R^M_{t+1} - \log R^F_{t+1}) + \epsilon_t \tag{59}
\]

where \(\beta = 0.48\), as estimated in Table 1, \(\rho = 0.97\), and \(\alpha\) is adjusted so that the long run growth of top wealth shares is zero.

While this dynamics for top wealth shares is misspecified (see Proposition 7), running this regression in the baseline model estimated in Table 5 gives a very accurate dynamics.\(^{47}\)

In contrast, Figure 4 shows that there are large, low-frequency fluctuations in top wealth shares that cannot be explained by asset returns. In particular, asset returns alone cannot explain the long term decline in inequality in the 1930s. Symmetrically, asset returns cannot fully explain the increase of inequality starting from 1980.

\(^{47}\)An alternative would be to construct the series of top wealth shares from the model, given the actual series of endowment shocks, but this combines how well the model fit the actual stock market return realization, and how well the stock market realization predicts top wealth shares.
Augmented Model  Formally, I modify the model by introducing a mean-reverting process \( \nu_t \) that fluctuates around zero:\(^{48}\)

\[
d\nu_t = -\kappa \nu_t dt + \sigma \nu dZ_t
\]

(60)

I assume that this process drives a wedge between the financial return available to the agents in group \( j \in \{A, B\} \) becomes:

\[
\mu_{At} = r_t + \delta + \kappa \sigma_{At} - c_{At} + (1 - x_t) \nu_t
\]

(61)

\[
\mu_{Bt} = r_t + \delta + \kappa \sigma_{Bt} - c_{Bt} - x_t \nu_t
\]

(62)

where \( \nu_t \) is a mean reverting process which fluctuates around zero. Note that the total return of wealth is unchanged by the process. The process \( \nu_t \) can be seen as a fluctuating wealth tax: the government levies a wealth tax \( \nu_t \) on agents in group \( A \) and redistribute the proceeds to all agents in proportion to their wealth. In other words, these shocks are purely redistributive. While I do not take a stand on the origin of these differential investment opportunities, the literature suggests some potential origins. These fluctuations could be generated by changes in technology (Garleanu et al. (2012), Kogan et al. (Forthcoming)), changes in financial frictions (Kiyotaki and Moore (1997), Di Tella (2016)), or changes in taxes (Piketty and Zucman (2015), Hubmer et al. (2016), Pastor and Veronesi (2016)).

These shocks introduce long-run changes in the growth rate of the wealth share of group \( A \), \( x_t \). Compared to Proposition 2, the law of motion of the wealth share becomes:

\[
\mu_{xt} = x_t (\mu_{W_{At}} - \mu - \mu_{pt} - \sigma \sigma_{pt}) - \sigma_{xt}(\sigma + \sigma_{pt}) + (\delta + n)(\pi_A \phi_t - x_t) - x_t(1 - x_t) \nu_t
\]

(63)

This generates additional dynamics in the wealth distribution compared to the baseline model.

Moreover, these shocks amplify the excess volatility of asset prices: in the augmented model, the price-dividend ratio of the representative firms reacts to news about the level of inequality \( x_t \), but also to news about the future growth of inequality \( \nu_t \):

\[
\sigma_R = \sigma + \left( \frac{\partial \log pd}{\partial x} \sigma_x + \left( \frac{\partial \log pd}{\partial \nu} \sigma_\nu \right) \right)
\]

(64)

The semi-elasticity of the price-dividend ratio to \( \nu \) is positive for two reasons. First, because agents in group \( A \) have a higher EIS than agents in group \( B \), a rise in \( \nu_t \) increases the aggregate demand.

\(^{48}\)When \( \sigma_\nu = 0 \), the model reverts to the baseline model.
for assets, which pushes up the price-dividend ratio. Second, a rise in $\nu_t$ also increases the growth rate of $x_t$, the share of wealth owned by the agents in group $A$, and is therefore associated with a lower interest rate and a lower market price of risk in the future. This further pushes up asset prices.

**Results.** I now estimate the model augmented with a mean reversing process $\nu_t$ with $\kappa_\nu = 0.05$ and $\sigma_\nu = 0.005$. This calibration corresponds to shocks with a half live of 14 years, which appears consistent with Figure 4. The long-run standard deviation of the process equals 1.5%, which is relatively small. As in the long-run risk literature (Bansal and Yaron (2004), small but persistent changes in investment opportunities can have large effects on asset prices.

The last column of Table 8 demonstrates that the augmented model can jointly match asset prices and the wealth distribution. In particular, the model can generate a high volatility of returns $\sigma_R = 17.2\%$ together with a high tail index $\zeta = 1.5$. In contrast, the baseline model could not jointly match these moments. The augmented model solves this tension through fluctuations in $\nu$. These fluctuations increase the volatility of returns, without changing the average wealth growth of top households $\mu_w A_t$, and therefore without increasing the right tail of the wealth distribution.

### 7 Conclusion

The results of this paper depict a strong interplay between asset prices and wealth inequality. Because rich households hold more risky assets, realized stock returns generate fluctuations in wealth inequality over time. Conversely, in periods of high inequality, more wealth is in the hands of rich households, the risk-tolerant investors. Therefore, risk premia are low: a high level of inequality predicts low future returns.

This interplay is at the heart of heterogeneous agents asset pricing models. I have shown that these models can qualitatively account for these facts. However, a simple model cannot generate the excess volatility of asset prices without overestimating the thickness of the tail of the wealth distribution. I suggest two parsimonious deviations from the baseline model. In the first, I explore the role of transitory preferences. In the second, I explore the role of differences in investment opportunities This approach provides an explanation for temporary disconnects between inequality and asset prices.

The implications of my analysis extend beyond asset pricing. The interplay I put forward can
have effects on real quantities as well, through two channels. First, because the level of inequality affects the cost of capital, this can lead to changes in corporate investment policies. Second, a recent literature has also emphasized the role of inequality for aggregate demand (Mian et al. (2013), Kaplan et al. (2016)). Exploring these channels requires moving away from an endowment economy, which I leave for future research.
Table 1: The Exposure to Stock Returns Increases Across the Wealth Distribution

<table>
<thead>
<tr>
<th>Flow of Funds</th>
<th>Kopczuk and Saez (2004)</th>
<th>Forbes 400</th>
</tr>
</thead>
<tbody>
<tr>
<td>All Households</td>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>Excess Stock Returns</td>
<td>0.44***</td>
<td>0.64***</td>
</tr>
<tr>
<td></td>
<td>(0.07)</td>
<td>(0.17)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.44</td>
<td>0.22</td>
</tr>
<tr>
<td>N</td>
<td>53</td>
<td>53</td>
</tr>
</tbody>
</table>

Panel B: Wealth Share

| Excess Stock Returns| 0.17***                  | 0.34***    | 0.48***    | 0.46**     |
|                     | (0.05)                   | (0.06)     | (0.13)     | (0.21)     |
| $R^2$               | 0.20                     | 0.39       | 0.23       | 0.13       |
| N                   | 53                       | 53         | 53         | 31         |

Notes: This table reports the results of the regression of the excess wealth growth of households in a given percentile group on the excess stock returns, i.e. Equation (1):

$$\log \left( \frac{W_{G,t+1}}{W_{G,t-1}} \right) - 4 \log R_{ft} = \alpha_G + \beta_G (\log R_{Mt} - \log R_{ft}) + \epsilon_{Gt}$$

The dependent variable is the growth of wealth in Panel A and the growth of wealth shares in Panel B. Each column corresponds to a different group of households. The first column corresponds to all U.S households. Columns (2) to (4) correspond to increasing top percentiles in the wealth distribution, using data from Kopczuk and Saez (2004). Columns (5) correspond to the Top 0.0003% ; the percentiles are chosen so that the group include the 400 wealthiest individuals in 2015. Estimation via OLS. Standard errors in parentheses and estimated using Newey-West with 4 lags. *, **, *** indicate significance at the 0.1, 0.05, 0.01 levels, respectively.
Table 2: The Exposure to Stock Returns Across the Wealth Distribution: Controlling for Composition

<table>
<thead>
<tr>
<th></th>
<th>Forbes 400</th>
<th></th>
<th></th>
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</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>All</td>
<td>Within</td>
</tr>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
<td></td>
</tr>
<tr>
<td>Excess Stock Returns</td>
<td>0.93***</td>
<td>0.89***</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.25)</td>
<td>(0.25)</td>
<td></td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.31</td>
<td>0.29</td>
<td></td>
</tr>
</tbody>
</table>

Notes: In the first column, the table regresses the total growth of the wealth in the top 400 on excess stock market returns, as in Table 1. In the second column, the table regresses the “within growth” on excess stock market returns. The “within growth” corresponds to the yearly growth of households in the top 400, whether or not they drop out of the top by the end of the year. See Gomez (2018) for more details on the construction of the “within” term.

Table 3: The Share of Wealth Owned by the Top 0.01% And Future Excess Returns

<table>
<thead>
<tr>
<th></th>
<th>$\sum_{1 \leq h \leq H} \log R_{t+h}^H - \log R_{t+h}^f = \alpha + \beta_H \text{Log Top Wealth Shares}<em>t + \gamma_H X_t + \epsilon</em>{tH}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Excess Returns at Horizon $H = 1$</td>
</tr>
<tr>
<td></td>
<td>Excess Returns at Horizon $H = 3$</td>
</tr>
<tr>
<td></td>
<td>$\beta_1$</td>
</tr>
<tr>
<td>Log Top Share</td>
<td>$-0.109^*$</td>
</tr>
<tr>
<td>$\Delta$ Log Top Share (5 years difference)</td>
<td>$-0.271^*$</td>
</tr>
<tr>
<td>Log Top Share and Linear Trend</td>
<td>$-0.264^{***}$</td>
</tr>
<tr>
<td>Log Top Share and Dividend Price</td>
<td>$-0.172^{**}$</td>
</tr>
<tr>
<td>Log Top Share and Dividend Payout</td>
<td>$-0.141^{**}$</td>
</tr>
</tbody>
</table>

Notes: The table reports the result of the regressions of future excess returns on the share of wealth owned by the Top 0.1% (row 1). Each row corresponds to a different set of regressors. Columns (1) (2) (3) report the results when the dependent variable is the one year excess-return. Columns (4) (5) (6) report the results when the dependent variable is the three-year excess-return. Estimation via OLS. Standard errors in parentheses and estimated using Newey-West with 4 lags. *, **, *** indicate significance at the 0.1, 0.05, 0.01 levels.
### Table 4: The Share of Wealth Owned by the Top 0.01% And Future Excess Returns
(Campbell and Yogo (2006) test)

<table>
<thead>
<tr>
<th></th>
<th>Confidence Interval for $\beta \in [\bar{\beta}, \bar{\beta}]$ in the Predictability Regression (2)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Case with $\rho = 0.89$</td>
</tr>
<tr>
<td>Log Top Share</td>
<td>$\bar{\beta}$</td>
</tr>
<tr>
<td></td>
<td>$-0.39$</td>
</tr>
<tr>
<td></td>
<td>$\bar{\beta}$</td>
</tr>
<tr>
<td></td>
<td>$-0.02$</td>
</tr>
</tbody>
</table>

**Notes:** The time period is 1917-1951, the longest period where the wealth share of the top 0.01% is available without missing years. This table uses the test developed Campbell and Yogo (2006), that jointly takes into account the persistence of the predictor as well as its correlation with stock returns to compute the 90% confidence interval for $\beta$. The autoregressive lag length for the DF-GLS statistic is estimated to be 1, using the Bayes information criterion (BIC).

### Table 5: Fitting the Baseline Model to the Data

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Baseline Model Estimated on:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Asset Prices</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1)</td>
</tr>
<tr>
<td>Parameters</td>
<td></td>
<td></td>
</tr>
<tr>
<td>RRA $\gamma_A$</td>
<td></td>
<td>1.5</td>
</tr>
<tr>
<td>RRA $\gamma_B$</td>
<td></td>
<td>17</td>
</tr>
<tr>
<td>EIS $\psi_A$</td>
<td></td>
<td>0.8</td>
</tr>
<tr>
<td>EIS $\psi_B$</td>
<td></td>
<td>0.05</td>
</tr>
<tr>
<td>Subjective Discount Rate $\rho$</td>
<td></td>
<td>0.1%</td>
</tr>
<tr>
<td>Population share $\pi_A$</td>
<td></td>
<td>1%</td>
</tr>
<tr>
<td>Moments</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Equity Premium</td>
<td></td>
<td>5.2%</td>
</tr>
<tr>
<td>STD Market Return</td>
<td></td>
<td>18.2%</td>
</tr>
<tr>
<td>Average interest rate</td>
<td></td>
<td>2.8%</td>
</tr>
<tr>
<td>STD interest rate</td>
<td></td>
<td>0.9%</td>
</tr>
<tr>
<td>Exposure Top Percentile $\beta$</td>
<td></td>
<td>0.95</td>
</tr>
<tr>
<td>Tail Index $\zeta$</td>
<td></td>
<td>1.5</td>
</tr>
<tr>
<td>$\beta_{log S_t \rightarrow R_{M_t+1} - R_{ft+1}}$</td>
<td></td>
<td>$-0.1$</td>
</tr>
<tr>
<td>Risk-Free Rate Channel %</td>
<td></td>
<td>25%</td>
</tr>
<tr>
<td>J-statistic</td>
<td></td>
<td>0.05</td>
</tr>
</tbody>
</table>

**Notes:** Column (1) corresponds to the moments in the data. Column (2) reports the corresponding moments in the model estimated on asset prices only. Column (3) reports the moments in the model estimated on asset prices and inequality moments. Numbers in bold indicate that these moments have been used in the estimation.
Table 6: Fitting the Baseline Model to the Data (Realistic Ingredients)

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Model</th>
<th></th>
<th># Discount Factors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td></td>
</tr>
<tr>
<td>Parameters</td>
<td>RRA $\gamma_A$</td>
<td>$1.4$</td>
<td>$1.5$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>RRA $\gamma_B$</td>
<td>$17$</td>
<td>$17$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>EIS $\psi_A$</td>
<td>$0.6$</td>
<td>$0.7$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>EIS $\psi_B$</td>
<td>$0.05$</td>
<td>$0.05$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Population share $\pi_A$</td>
<td>$11.2%$</td>
<td>$9.0%$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Discount Rate $\rho_A$</td>
<td>$0.1%$</td>
<td>$0.1%$</td>
<td>$2%$</td>
</tr>
<tr>
<td>Moments</td>
<td>Equity Premium</td>
<td>$5.2%$</td>
<td>$3.4%$</td>
<td>$3.6%$</td>
</tr>
<tr>
<td></td>
<td>STD Market Return</td>
<td>$18.2%$</td>
<td>$16.3%$</td>
<td>$16.2%$</td>
</tr>
<tr>
<td></td>
<td>Average interest rate</td>
<td>$2.8%$</td>
<td>$3.7%$</td>
<td>$3.1%$</td>
</tr>
<tr>
<td></td>
<td>STD interest rate</td>
<td>$0.9%$</td>
<td>$1.0%$</td>
<td>$0.8%$</td>
</tr>
<tr>
<td></td>
<td>Exposure Top Percentile $\beta$</td>
<td>$0.95$</td>
<td>$0.96$</td>
<td>$1.0$</td>
</tr>
<tr>
<td></td>
<td>Tail Index $\zeta$</td>
<td>$1.5$</td>
<td>$1.5$</td>
<td>$1.5$</td>
</tr>
<tr>
<td>$\beta_{\log S_t \rightarrow R_{Mt+1} - R_{ft+1}}$</td>
<td>$-0.1$</td>
<td>$-0.03$</td>
<td>$-0.06$</td>
<td></td>
</tr>
<tr>
<td>Risk-Free Rate Channel %</td>
<td>$56%$</td>
<td>$49%$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Notes: Column (1) corresponds to the moments in the data. Columns (2) to (3) correspond to different estimations of the extended model. All models are estimated on six moments: four moments about asset prices, and two moments about the wealth distribution (the exposure of top wealth shares to stock market returns and the tail index). Column (2) reports the model estimated with idiosyncratic volatility $\nu \approx 10\%$. Column (3) reports the model estimated with heterogeneous subjective discount rates.
Table 7: Exposure to Stock Market Returns Within Forbes 400

<table>
<thead>
<tr>
<th>Exposure to Stock Market Returns $\beta$</th>
<th>(1)</th>
<th>(2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Excess Stock Returns</td>
<td>1.02***</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.25)</td>
<td></td>
</tr>
<tr>
<td>Excess Stock Returns $\times {\text{Age} \geq \text{Median}}$</td>
<td>−0.31</td>
<td>−0.31***</td>
</tr>
<tr>
<td></td>
<td>(0.36)</td>
<td>(0.12)</td>
</tr>
<tr>
<td>Fixed Effects</td>
<td>Year</td>
<td></td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.31</td>
<td>0.92</td>
</tr>
<tr>
<td>Period</td>
<td>1983-2013</td>
<td>1983-2013</td>
</tr>
<tr>
<td>$N$</td>
<td>62</td>
<td>62</td>
</tr>
</tbody>
</table>

Notes: Every year, I construct the (wealth-weighted) average return of households in Forbes 400 within two groups: those with age below the median ($g = 1$), and those with age above the median ($g = 2$). The median corresponds to 60 years. Denote $R_{gt}$ the return of households in group $g$ at year $t$. Column (1) reports the results of estimating the model:

$$\sum_{t=1}^{4} \log R_{gt} - 4 \log R_{ft} = \alpha + \delta I_{\text{Age} \geq \text{Median}} + \beta (\log R_{Mt} - \log R_{ft}) + \gamma I_{\text{Age} \geq \text{Median}} (\log R_{Mt} - \log R_{ft}) + \epsilon_{gt}$$

A negative $\gamma$ says that the stock market exposure of old households is lower than the stock market exposure of young households. The four year horizon follows the specifications in Table 1. Column (2) adds year fixed effects. Estimation via OLS. Standard errors in parentheses and estimated using Newey-West with 4 lags. *, **, *** indicate significance at the 0.1, 0.05, 0.01 levels.
Table 8: Fitting the Augmented Model to the Data

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Augmented Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td><strong>Calibration</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Transition Rate $\tau$</td>
<td>0.02</td>
<td></td>
</tr>
<tr>
<td>Persistence $\kappa_x$</td>
<td></td>
<td>0.05</td>
</tr>
<tr>
<td>Volatility $\sigma_x$</td>
<td></td>
<td>0.006</td>
</tr>
<tr>
<td><strong>Parameters</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>RRA $\gamma_A$</td>
<td>1.2</td>
<td>2.0</td>
</tr>
<tr>
<td>RRA $\gamma_B$</td>
<td>18</td>
<td>17</td>
</tr>
<tr>
<td>EIS $\psi_A$</td>
<td>1.0</td>
<td>0.6</td>
</tr>
<tr>
<td>EIS $\psi_B$</td>
<td>0.05</td>
<td>0.06</td>
</tr>
<tr>
<td>Population share $\pi_A$</td>
<td>0.09</td>
<td>0.08</td>
</tr>
<tr>
<td>Discount Rate $\rho$</td>
<td>1.0%</td>
<td>0.1%</td>
</tr>
<tr>
<td><strong>Moments</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Equity Premium</td>
<td>5.2%</td>
<td>5.1%</td>
</tr>
<tr>
<td>STD Market Return</td>
<td>18.2%</td>
<td>18.3%</td>
</tr>
<tr>
<td>Average interest rate</td>
<td>2.8%</td>
<td>2.8%</td>
</tr>
<tr>
<td>STD interest rate</td>
<td>0.9%</td>
<td>0.7%</td>
</tr>
<tr>
<td>Exposure Top Percentile $\beta$</td>
<td>0.95</td>
<td>0.90</td>
</tr>
<tr>
<td>Tail Index $\zeta$</td>
<td>1.5</td>
<td>1.5</td>
</tr>
<tr>
<td>$\beta_{log} s_t \rightarrow R_{M_t+1} - R_{f_{t+1}}$</td>
<td>-0.1</td>
<td>-0.08</td>
</tr>
<tr>
<td>Risk-Free Rate Channel %</td>
<td>30%</td>
<td>45%</td>
</tr>
<tr>
<td>J-statistic</td>
<td>0.15</td>
<td>1.6</td>
</tr>
</tbody>
</table>

Notes: Column (1) corresponds to the moments in the data. Columns (2) corresponds to estimations of the model with transition rate. All models are estimated on six moments: four moments about asset prices, and two moments about the wealth distribution (the exposure of top wealth shares to stock market returns and the tail index).
Figure 1: Exposure $\beta$ of Top Wealth Group to Stock Market Returns at Different Horizons

(a) Estate Tax Returns

(b) Forbes 400

Notes: This figure reports the results of regressing excess wealth growth of households in the top 0.01% (left) and in the top 400 (right) on excess stock returns i.e.

$$\log \left( \frac{W_{G,t-1+h}}{W_{G,t-1}} \right) - h \log R_{ft} = \alpha_{Gh} + \beta_{Gh}(\log R_{Mt} - \log R_{ft}) + \epsilon_{Ght}$$

Each figure reports the estimates for $\beta_{Gh}$, from $h = 0$ to $h = 8$, as well as the 5%-95% confidence interval using Newey-West with 4 lags.
Figure 2: The Right Tail of the Wealth Distribution: Data vs Baseline Model

Notes: The figure compares the log networth (relative to the average networth) to the log percentile in SCF and Forbes. More precisely, the figure plots the average log networth within 40 logarithmically spaced percentile bins in SCF. The figure plots the average log networth for each position in Forbes 400. The (opposite of) the slope estimate gives $\zeta \approx 1.5$ for SCF and for Forbes 400. In red, the figure plots the distribution implied by the model estimated on asset prices (see Table 5).
Figure 3: Asset Prices in the Baseline Model

(a) EIS and RRA

(b) Interest Rate and Market Price of Risk

(c) Stationary Density

(d) Campbell-Shiller Decomposition

Notes: The figure plots the interest rate, the market price of risk, the stationary density, and the Campbell-Shiller Decomposition of Proposition 3 for the model estimated on asset prices only (Columns (2) in Table 5). The Campbell-Shiller Decomposition is estimated by constructing $E[\gamma|\mathbf{x}_0 = x]$ and $E[\kappa_t\sigma_{R_t} - \frac{1}{2}\sigma_{R_t}^2|\mathbf{x}_0 = x]$ using Kolmogorov-Backward equation.
Figure 4: Time Series of the Top 0.01% vs Top 0.01%

[Graph showing time series data for different sources: Top 0.01% (Kopczuk Saez), Top 0.01% (Forbes 400), Top 0.01% (Saez Zucman), Top 0.01% (Imputed).]
Appendix A

A.1 Cross Sectional Evidence

To understand better what drives the heterogeneous exposure of top households to aggregate shocks, I also examine the heterogeneity in equity holding across the wealth distribution using the Survey of Consumer Finances (SCF). The survey is a repeated cross-section of about 4,000 households per survey year, including a high-wealth sample. The survey is conducted every three years, from 1989 to 2013. The respondents provide information on their networth, including their investment in public and private equity. I define the equity share as the total investment in equity over networth. I define the set of entrepreneurs as the households with equity held in an actively managed business.49

Figure A1a plots the average equity share within percentile bins across the wealth distribution. The average equity share of 0.4 masks a substantial heterogeneity across households. The equity share is essentially flat at 0.2 over the majority of the wealth distribution, but increases sharply within the top 1%. Figure A1b plots the equity share with respect to the log top percentiles, showing that the equity share is approximately linear in the log percentile at the top of the distribution. The figure suggests that the bulk of the heterogeneity is concentrated within the top percentiles, which justifies my focus on the top percentile.

A stylized fact in the household finance literature is that stock market participation increases with wealth (Vissing-Jørgensen (2002)). Therefore, the increase in the equity share within the top percentiles could be driven by an increase in the proportion of stockholders (i.e. the extensive margin). However, Panel B of Table A1 shows that the percentage of stockholders is constant within the top percentiles (90%). The increase in the equity share is entirely driven by the increase within stockholders. While the heterogeneity between stockholders and non-stockholders generates a lot of variations at the bottom of the wealth distribution, these variations account for a small share of total wealth.

Investment in risky assets comes mainly in two forms: public equity and private equity. Panel A of Table A1 decomposes the increase in equity share across the top percentiles between the two types of equity. The decomposition reveals that the increase in the equity share is mostly driven by an increase in the share of wealth invested in private equity. Panel C of Table A1 shows that the proportion of entrepreneurs increases sharply in the top percentile: the proportion of households with an actively managed business is 78.5% in the Top 0.01%, compared to 10% in the general population.50 The wealth of these entrepreneurs is mostly invested in their private business. A potential concern is that, if entrepreneurs cannot trade or sell their firms easily, the heterogeneity in private equity holdings may have no impact on stock market prices. However, Panel C of Table A1 shows that entrepreneurs hold substantial amounts of public equity (15%). Even with illiquid businesses, entrepreneurs can adjust their overall risky holdings at the margin.

49The definition follows Moskowitz and Vissing-Jørgensen (2002).
50Similarly, Hurst and Lusardi (2004), using the Panel Study of Income Dynamics (PSID), show that the propensity of entrepreneurship increases sharply with wealth in the top percentiles.
A.2 Saez-Zucman series

Saez and Zucman (2016) have recently proposed a new series for top wealth shares, which relies on Income Tax Returns. In Table A3, I estimate the stock market exposure of top wealth percentiles using this series. Qualitatively, the results of Section 2 hold true: the exposure of top wealth shares to stock market return increases with top percentile. However, the estimates are now uniformly lower compared to Kopczuk and Saez (2004). For instance, the stock market exposure of the Top 0.01% is 0.66 using Income Tax Returns, compared to 0.95 using estate tax returns or Forbes. This suggests that the methodology used in Saez and Zucman (2016) cannot capture business cycle frequencies of wealth shares, even though they may capture more accurately the long run fluctuations in inequality. Indeed, a certain number of wealth categories are constructed using trends and interpolations across years, which tends to bias down the estimates.

A.3 Composition Change

As shown in Table 2, changes in composition do not matter in driving the exposure of top wealth shares to stock market returns. As proven in Gomez (2018), the role of composition changes for the growth rate of top shares is well approximated by \((\zeta - 1)\nu^2/2\), where \(\zeta\) denotes the power-law exponent \(\zeta\) of the wealth distribution and \(\nu^2\) denotes the idiosyncratic variance of wealth at the top.

This formula allows me to express the difference between the exposure of top wealth shares and the exposure of individual households at the top (i.e. the estimates in Column(1) and (2) Table 2):

\[
\frac{\text{cov}(\zeta - 1, \log R_{mt})}{\text{var} \log R_{mt}} \approx \frac{\zeta - 1}{2} \frac{\text{std}(\nu^2)}{\text{std}(\log R_{mt})} \rho_{\nu^2, \log R_{mt}}
\]

This difference is the product of three terms (i) a term that depends on the shape of the wealth distribution \(\frac{1}{2}(\zeta - 1)\) (ii) a term that depends on the ratio between the standard deviation of idiosyncratic volatility compared to the standard deviation of the stock market return (iii) the correlation between idiosyncratic volatility and stock market returns. In other words, because the wealth distribution is very concentrated (\(\zeta\) is close to one), and because yearly fluctuations in idiosyncratic volatility are small compared to the yearly fluctuations in stock market return, we can expect the difference to be small.

Appendix B

B.1 Proof

Proof of Proposition 1. The HJB equation associated with household’s problem is

\[
0 = \max_{c_{jt}, s_{W_{jt}}} \{f(C_{jt}, V_{jt}) + E[dV_{jt}]\}
\]

(A1)
Given the homotheticity assumptions, the value function of the households in group \( j \in \{A, B\} \) with wealth \( W \) can be written:

\[
V_{jt}(W) = \frac{W^{1-\gamma_j}}{1-\gamma_j} p_{jt}^{\frac{1-\gamma_j}{\psi_j}}
\]  
(A2)

Applying Ito’s lemma on HJB equation:

\[
0 = \max_{C_{jt}, \sigma_{W_{jt}}} \left\{ \frac{1}{1-\frac{1}{\psi_j}} \left( \frac{C_{jt}^{1-\frac{1}{\psi_j}}}{W_{jt}^{1-\frac{1}{\psi_j}} \sigma_{W_{jt}}^2} - (\rho + \delta) \right) + (1-\gamma_j)\mu_{W_{jt}} + \frac{\gamma_j(\gamma_j-1)}{2} \sigma_{W_{jt}}^2 \right. \\
+ \left. \frac{1-\gamma_j}{\psi_j-1} \mu_{p_{jt}} + \frac{1-\gamma_j}{2(\psi_j-1)} \left( \frac{1-\gamma_j}{\psi_j-1} - 1 \right) \sigma_{p_{jt}}^2 + \frac{\gamma_j}{\psi_j-1} \sigma_{W_{jt}} \sigma_{p_{jt}} \right\}
\]  
(A3)

Substituting the expression for the wealth drift \( \mu_j \) using the budget constraint and dividing by \( 1-\gamma_j \)

\[
0 = \max_{C_{jt}, \sigma_{W_{jt}}} \left\{ \frac{1}{1-\frac{1}{\psi_j}} \left( \frac{C_{jt}^{1-\frac{1}{\psi_j}}}{W_{jt}^{1-\frac{1}{\psi_j}} \sigma_{W_{jt}}^2} - (\rho + \delta) \right) + r_t + \delta + \sigma_{W_{jt}} \kappa_t - \frac{C_{jt}}{W_{jt}} - \frac{\gamma_j}{2} \sigma_{p_{jt}}^2 \\
+ \frac{1}{\psi_j-1} \mu_{p_{jt}} + \frac{1}{2(\psi_j-1)} \left( \frac{1-\gamma_j}{\psi_j-1} - 1 \right) \sigma_{p_{jt}}^2 + \frac{1-\gamma_j}{\psi_j-1} \sigma_{W_{jt}} \sigma_{p_{jt}} \right\}
\]  
(A4)

The FOC for aggregate risk exposure gives

\[
\sigma_{W_{jt}} = \frac{\kappa_t}{\gamma_j} + \frac{1}{\psi_j-1} \sigma_{p_{jt}}
\]  
(A5)

The FOC for consumption gives

\[
C_{jt} = \frac{1}{p_{jt} W_{jt}}
\]  
(A6)

that is, \( p_{jt} \) is the wealth / consumption ratio of the household.

Plugging the optimal consumption rate into the HJB, we obtain an expression for the wealth drift in terms of \( r_t \) and \( \kappa_t \):

\[
\mu_{W_{jt}} = r_t + \delta + \sigma_{W_{jt}} \kappa_t - \frac{1}{p_{jt}} \\
= \psi_j (r - \rho) \psi_j + \frac{1}{2} \psi_j \kappa_t \sigma_{p_{jt}} + \frac{\psi_j - \psi_j - 1}{(\psi_j - 1)^2} \sigma_{p_{jt}}^2 + \mu_{p_{jt}}
\]  
(A7)

**Proof of Proposition 3.** I derive a Campbell-Shiller decomposition in continuous-time. The cumulative return of a dollar invested in the asset follows the process:

\[
\frac{dR_t}{R_t} = \frac{1}{p_{Dt}} + \frac{d(p_{Dt} D_t)}{p_{Dt} D_t}
\]  
(A8)

Therefore, in log,

\[
d \log R_t = \frac{1}{p_{Dt}} dt + d \log p_{Dt} + d \log D_t
\]  
(A9)
As in Chacko and Viceira (2005), I log-linearize the dividend-price ratio around the average value of log $p_D$:

$$\frac{1}{p_D} \approx \alpha(1 - \log \alpha) - \alpha \log p_D t$$

(A10)

with $\alpha = e^{-E[\log p_D]}$. Plugging this expression into Equation (A9), I obtain:

$$d \log R_t \approx \alpha(1 - \log \alpha)dt - \alpha \log p_D t dt + d \log p_D t + d \log D_t$$

(A11)

Integrating, the price-dividend ratio can be written as:

$$\log p_D t \approx E_t \left[ \int_t^{+\infty} e^{-\alpha(s-t)} (\alpha(1 - \log \alpha)dt + d \log R_s - d \log D_s) \right] \approx 1 - \log \alpha + E_t \left[ \int_t^{+\infty} e^{-\alpha(s-t)} (d \log D_s - d \log R_s) \right]$$

(A12)

Proof of Proposition 2. Applying Ito’s lemma, the law of motion of the wealth per capita in group $A$ is:

$$d \left( \int_{t \in A_t} W_{it}/(\pi_A N_t) \right) = d \left( \int_t^t (\delta + n) e^{-(\delta+n)(t-s)} W_{Ats} ds \right)$$

$$= \int_t^t (\delta + n) e^{-(\delta+n)(t-s)} dW_{Ats} ds + \pi_A N_t (\delta + n) W_{At} dt$$

$$- (\delta + n)dt \int_{-\infty}^t (\delta + n) e^{-(\delta+n)(t-s)} W_{Ats} ds$$

(A14)

Dividing by $\int_{t \in A_t} W_{it}/(\pi_A N_t)$ we obtain

$$\frac{d \left( \int_{t \in A_t} W_{it}/(\pi_A N_t) \right)}{\int_{t \in A_t} W_{it}/(\pi_A N_t)} = \frac{dW_{Ats}}{W_{Ats}} + (\delta + n) \left( \phi_t \frac{\pi_A}{x_t} - 1 \right) dt$$

(A15)

This gives the law of motion of $x_t$.

B.2 Solving the Model

Assume that $G(u)$ is a sum of $K$ exponential

$$G_k(u) = B_k e^{-\delta_k u} \forall 1 \leq k \leq K$$

(A16)

$$G(u) = \sum_{1 \leq k \leq K} G_k(u)$$

(A17)
where the coefficients \((B_k)_{1 \leq k \leq K}\) are such that total aggregate earnings equal \(\omega Y_t\)

\[
1 = \sum_{1 \leq k \leq K} B_k \frac{\delta + n}{\delta + n + \delta_k}
\]  
(A18)

Demote \(p^l_k\) as the price-dividend of a claim with exponentially decreasing endowment at rate \(\delta + \delta_k\), for \(1 \leq k \leq K\). Conjecture that this process follows a diffusion process

\[
\frac{dp^l_k}{p_k} = \mu p^l_k dt + \sigma_p p^l_k dZ_t
\]  
(A19)

Using Equation (10), we obtain

\[
\phi = \omega \sum_{1 \leq k \leq K} B_k \frac{p^l_k}{p}
\]  
(A20)

We also note that

\[
\frac{1}{p} = \frac{x}{p_A} + \frac{1 - x}{p_B}
\]  
(A21)

gives the first and second derivative of \(p\) with respect to \(x\) in terms of the first and second derivatives of \(p_A\) and \(p_B\) with respect to \(x\)

**Solve for \(\sigma_x\)** Applying Ito’s lemma, we have

\[
\sigma_{p_j} = \frac{\partial_x p_j}{p_j} \sigma_x \quad \text{for } j \in \{A, B\}
\]  
(A22)

\[
\sigma_p = \frac{\partial_x p}{p} \sigma_x
\]  
(A23)

\[
\sigma_{p^l_k} = \frac{\partial_x p^l_k}{p^l_k} \sigma_x \quad \text{for } l \in 1 \leq k \leq K
\]  
(A24)

Substituting the expression for \(\kappa\) in Equation (26) in Proposition 2, we can solve for \(\sigma_x:\)

\[
\sigma_x = \sigma_p \frac{\frac{x(1-x)\Gamma}{\gamma_A \gamma_B} (\gamma_B - \gamma_A) \sigma}{1 - \frac{x(1-x)\Gamma}{\gamma_A \gamma_B} \frac{\gamma_A - \gamma_B}{\gamma_A + 1} \psi_A - 1 \frac{\gamma_A - \gamma_B}{\gamma_B - 1} \psi_B - 1 \frac{\gamma_A - \gamma_B}{\gamma_B - 1} \psi_B - 1}
\]  
(A25)

**Solve for \(\mu_x\)** The law of motion of \(x\) is

\[
dx_t = \mu_xt dt + \sigma_xt dZ_t
\]  
(A26)

where \(\mu_xt\) and \(\sigma_xt\) are given by

\[
\sigma_xt = x_t (\sigma_{W_At} - \sigma - \sigma_{p_t}),
\]  
(A27)

\[
\mu_xt = x_t (\mu_{W_At} - \mu - \mu_{p_t} - \sigma_{p_t}) - \sigma_xt (\sigma + \sigma_{p_t}) + (\delta + n)(\pi_A \phi_t - x_t)
\]  
(A28)

In terms of previously computed quantities:

\[
\mu_x = x(1 - x) \left( \kappa (\sigma_{W_At} - \sigma_{W_Bt}) - \frac{1}{p_A} + \frac{1}{p_B} \right) + (\delta + n)\phi(x - \pi_A) + x\sigma^2_{p_t,Y_t} - x\sigma_{W_At}\sigma_{p_t,Y_t}
\]  
(A29)
Using Ito’s lemma, we can express the drift of all quantities to solve for:

\[ \mu_{p_j} = \frac{\partial_x p_j \mu_x}{p_j} + \frac{1}{2} \frac{\partial_{xx} p_j \sigma_x^2}{p_j} \text{ for } j \in \{A, B\} \]  

\[ \mu_p = \frac{\partial_x p \mu_x}{p} + \frac{1}{2} \frac{\partial_{xx} p \sigma_x^2}{p} \]  

\[ \mu_{p_l^k} = \frac{\partial_x p_l^k \mu_x}{p_l^k} + \frac{1}{2} \frac{\partial_{xx} p_l^k \sigma_x^2}{p_l^k} \]  

**System of ODEs** After obtaining the interest rate with Equation (30), the budget constraint for \(A, B\) and the definition for \((p_l^k)_{1 \leq k \leq K}\) gives a system of \(2 + K\) ODEs:

\[ \frac{1}{p_j} + \mu_{W_j} = r + \delta + \kappa \sigma_{W_j} \text{ for } j \in \{A, B\} \]  

\[ \frac{1}{p_l^k} + \mu - \delta_k + \sigma_{p_l^k} = r + \delta + \kappa (\sigma + \sigma_{p_l^k}) \text{ for } 1 \leq k \leq K \]  

where \(\mu_{W_j}\) and \(\sigma_{W_j}\) are given by Proposition 1.

**Price-Dividend Ratio** The ratio of total human capital to the total labor income at a given point in time is

\[ p_h = \sum_{1 \leq k \leq K} B_k \frac{\delta + n}{\delta + n + \delta_k} \phi_k \]  

Therefore, the price-dividend ratio \(pd\) for the representative firm is given by

\[ pd = \frac{1}{1 - \omega} (p - \omega p_h) \]  

Using Ito’s lemma, one can then obtain the return dynamics of the representative levered firm, as

\[ \frac{dR_t}{R_t} = (r + \kappa \lambda (\sigma + \sigma_{pd}) \sigma + \sigma_{pd}) dt + (\sigma + \sigma_{pd}) dZ_t \]

**B.3 Computational Method**

The system of PDEs is written on a state space grid and derivatives are substituted by finite difference approximations. Importantly, first order derivatives are upwinded.

Denote \(Y\) the solution and denote \(F(Y)\) the finite difference scheme corresponding to a model. The goal is to find \(Y\) such that \(F(Y) = 0\). I solve for \(Y\) using a fully implicit Euler method. Updates take the form

\[ \forall t \leq T \quad 0 = F(y_{t+1}) - \frac{1}{\Delta} (y_{t+1} - y_t) \]  

Each time step requires to solve a non-linear equation. I solve this non-linear equation using a Newton-Raphson method. The Newton-Raphson method converges if the initial guess is close enough to the solution. Since \(y_t\) converges towards \(y_{t+1}\) as \(\Delta\) tends to zero, one can always choose \(\Delta\) low enough so that the inner
steps converge. Therefore, I adjust $\Delta$ as follows. If the inner iteration does not converge, I decrease $\Delta$. If the inner iteration converges, I increase $\Delta$. After a few successful implicit time steps, $\Delta$ is large and, therefore, the algorithm becomes like Newton-Raphson. In particular, the convergence is quadratic around the solution.

This method is most similar to a method used in the fluid dynamics literature, called the Pseudo-Transient Continuation method. Formal conditions for the convergence of the algorithm are given in Kelley and Keyes (1998). The algorithm with $I = 1$ and $\Delta$ constant corresponds to Achdou et al. (2016). Allowing $I > 1$ and adjusting $\Delta$ are important to ensure convergence in the case on non-linear PDEs.

**Appendix C**

**Lemma 1** (Kolmogorov-Forward with Aggregate Risk). Suppose $w_t$ is a process evolving according to

$$\frac{dw_t}{w_t} = \mu_t dt + \sigma_t dZ_t$$

(A38)

where $Z_t$ is a standard aggregate Brownian Motion. Suppose that agents die with Poisson rate $\delta$ and, are born according to the density $g_{\chi t}$. Finally suppose that population grows with rate $n$. The density of $w_t$, $g_t$, follows the law of motion:

$$dg_t = \left(-\mu_t \partial_w(wg_t) + \frac{1}{2} \sigma_t^2 \partial_{ww}(w^2 g_t) + (\delta + n)(g_{\chi t} - g_t)\right) dt - \sigma_t \partial_w(wg_t) dZ_t$$

(A39)

**Proof of Lemma 1.** For any function $f$, we have

$$\int_{-\infty}^{+\infty} f(w)g_t + dt(w) dw = \int_{-\infty}^{+\infty} ((f(w) + df(w)) g_t(w) + f(w)(\delta + n) dt(g_{\chi t}(w) - g_t(w)) dw$$

(A40)

Assuming that $f$ is twice differentiable, Ito’s lemma gives:

$$\int_{-\infty}^{+\infty} f(w) dg_t(w) dw = \int_{-\infty}^{+\infty} \left(\mu_t w dt f(w) + \frac{1}{2} \sigma_t^2 w^2 dt \partial_{ww} f(w) + \sigma_t w \partial_w f(w) dZ_t\right) g_t(w) dw$$

$$+ \int_{-\infty}^{+\infty} f(w)(\delta + n) dt(g_{\chi t}(w) - g_t(w)) dw$$

(A41)

Assume that $f$ decays fast enough as $|x| \to +\infty$ and use integration by parts to obtain

$$\int_{-\infty}^{+\infty} f(w) dg_t(w) dw = \int_{-\infty}^{+\infty} f(w) \left(-\mu_t \partial_w(wg_t) + \frac{1}{2} \sigma_t^2 \partial_{ww}(w^2 g_t) + (\delta + n)(g_{\chi t} - g_t)\right) dw$$

$$- \int_{-\infty}^{+\infty} f(w) \sigma_t \partial_w(wg_t) dZ_t dw$$

(A42)

This equality must hold for all $f$ satisfying the conditions above. Therefore, we obtain

$$dg_t = \left(-\mu_t \partial_w(wg_t) + \frac{1}{2} (\sigma_t^2 + \nu_t^2) \partial_{ww}(w^2 g_t) + (\delta + n)(g_{\chi t} - g_t)\right) dt - \sigma_t \partial_w(wg_t) dZ_t$$

(A43)
Proof of Proposition 4. The proposition follows from Lemma 1 applied to the particular model.

Lemma 2 (Stability of Linear Functional). Let $x_t \in \mathbb{R}$ a continuous-time strong Markov process non explosive, irreducible, positive recurrent, with unique invariant probability measure.

Consider the process

$$dM_t = (\mu(x_t)M_t + b(x_t))dt + \sigma(x_t)M_tdW_t$$

(A44)

with $P(b(x) \geq 0) = 1$ and $P(b(x) > 0) > 0$.

(i) If $E[\mu(x) - \frac{1}{2}\sigma(x)^2] < 0$, the process converges to infinity a.s.
(ii) If $E[\mu(x) - \frac{1}{2}\sigma(x)^2] < 0$, the process does not converge to infinity a.s.

where $E$ denotes the expectation with respect to the invariant probability measure of $x$.

Proof of Lemma 2. The proof for the case $b(x_t) = 0$ (purely multiplicative process) is in Maruyama and Tanaka (1959). To my knowledge, however, there exists no proof for the more general process (A44), so I extend their proof to handle the case $b(x_t) \neq 0$. The idea is to bound the continuous time process by a discrete time process, which can be examined using Vervaat (1979).

For $\tau > 0$, we have the following recurrence equation:

$$M_{t+\tau} = e^{\int_0^{t+\tau} ((\mu(x_u) - \frac{1}{2}\sigma(x_u)^2)du + \sigma(x_u)dW_u)}M_t + \int_t^{t+\tau} e^{\int_s^{s+\tau} ((\mu(x_u) - \frac{1}{2}\sigma(x_u)^2)du + \sigma(x_u)dW_u)}b(x_s)ds$$

Denote $I$ the set of values that $x_1$ can take. Take $a < b$, both in $I$. Define the sequence of stopping times $S_0 = 0$ and

$$T_n \equiv \inf\{t > S_n; x_t = a\}$$
$$S_{n+1} \equiv \inf\{t > T_n; x_t = b\}$$

Define

$$Y_n = M_{T_n}$$
$$A_n = \exp\left(\int_{T_n}^{T_{n+1}} ((\mu(x_u) - \frac{1}{2}\sigma(x_u)^2)du + \sigma(x_u)dW_u)\right)$$
$$B_n = \int_{T_n}^{T_{n+1}} e^{\int_s^{T_{n+1}} ((\mu(x_u) - \frac{1}{2}\sigma(x_u)^2)du + \sigma(x_u)dW_u)}b(x_s)ds$$

The sequence $Y_n$ satisfies the following recurrence relation:

$$Y_{n+1} = A_nY_n + B_n$$

where $A_n$ and $B_n$ are i.i.d over time. Moreover, $A_1$ is positive a.s., $B_1$ is non negative a.s. with $P(B_1 > 0) > 0$ and $E[\log(B_1)] < +\infty$. As proven by Vervaat (1979), $Y_n$ converges in distribution if $E[\log A_1] < 0$, i.e.
\[
E \left[ \int_{T_1}^{T_2} (\mu(x_u) - \frac{1}{2} \sigma(x_u)^2) \, du \right] \geq 0, \text{ and converges a.s. to infinity if } E[\log A_1] > 0. \text{ Finally, as shown in Maruyama and Tanaka (1959), any integrable function } f, \ E \left[ \int_{T_1}^{T_2} f(x_u) \, du \right] \geq 0 \text{ iff } E[f(x)] \geq 0. \text{ I conclude by noting that } M_t \text{ converges to infinity a.s. if and only if } Y_n \text{ converges to infinity a.s. because, for } t \in (T_n, T_{n+1}], \ Y_n \leq M_t \leq Y_{n+1}. \]

\[\square\]

**Lemma 3** (Tail index). Let \( x_t \in \mathbb{R} \) a continuous-time strong Markov process non explosive, irreducible, positive recurrent, with unique invariant probability measure.

Suppose the dynamics of individual wealth \( w_{it} \) has the following law of motion:

\[
dw_{it} = \mu(x_t) dt + \sigma(x_t) dZ_t + \nu(x_t) dW_{it} \tag{A45}
\]

where \( W_{it} \) is an idiosyncratic Brownian Motion and \( Z_t \) is an aggregate Brownian Motion. Moreover, assume that individuals die with death rate \( \delta > 0 \) and are re-injected according to a distribution \( g_{xt} \) with thin tails.\(^{51}\)

If \( E[\nu(x)^2] > 0 \) or \( E[\mu(x)] - \frac{1}{2} E[\sigma(x)^2] > 0 \), the wealth distribution is fat-tailed with tail index \( \zeta \), given by the positive root of

\[
\zeta E[\mu(x)] - \frac{1}{2} E[\sigma(x)^2] + \frac{\zeta(\zeta - 1)}{2} E[\nu(x)^2] - \delta = 0 \tag{A46}
\]

where \( E \) denotes the expectation with respect to the stationary density of \( x \).

**Proof of Lemma 3.** Denote \( m_i^\xi \) the \( \xi \)-moment of the wealth distribution and \( m_{xt}^\xi \) the \( \xi \) cross-sectional moment of the distribution of wealth at birth. The law of motion of \( m_i^\xi \) is given by:

\[
dm_i^\xi = \left( \xi \mu(x_i) + \left( \frac{\xi(\xi - 1)}{2} \right) (\sigma(x_i)^2 + \nu(x_i)^2) \right) m_i^\xi dt + (\delta + n)(m_{xt}^\xi - m_i^\xi) dt + \xi \sigma(x_i) m_i^\xi dZ_i \tag{A47}
\]

Compared to the dynamics of the wealth density, the dynamics of \( m_{xt}^\xi \) can be examined in isolation for each \( \xi \), since it does not depend on the derivative of the function \( \xi \rightarrow m_{xt}^\xi \). This insight is due to Gabaix et al. (2016) and Luttmer (2012), that show that, in setups where wealth dynamics are linear in wealth, it is often easier to work with higher-order moments of wealth.

Denote \( f \) the function defined as:

\[
f(\xi) = \xi \left( E[\mu(x)] - \frac{1}{2} E[\sigma(x)^2] \right) + \frac{\xi(\xi - 1)}{2} E[\nu(x)^2] - (\delta + n) \tag{A48}
\]

There exists a unique \( \zeta > 0 \), such that \( f(\zeta) = 0 \). Moreover, \( f(\xi) \) is negative for \( \xi \in [0, \zeta) \) and positive for \( \xi \in (\zeta, +\infty) \). Using Lemma 2, we have that \( \zeta = \inf \{ \xi \in \mathbb{R}_+ | \lim_{t \rightarrow +\infty} m_i^\xi = +\infty \text{ a.s.} \}. \]

\[\square\]

**Proof of Proposition 5.** The first part of the proposition follows from Lemma 3 applied to the particular model.

\(^{51}\)Formally, the \( \xi \)th moments of \( \psi \) exists for \( \xi \leq \zeta. \)
I now prove that the distribution of financial wealth has the same tail index $\zeta$. For an individual with wealth $W_{it} \geq 0$, define $H_{it} \geq 0$ the capitalized value of his human capital, and $A_{it} = W_{it} - H_{it}$. Denote $A_{it}^+ = \max(A_{it}, 0)$.

Because $x \to x^\zeta$ is an increasing function, we have the lower bound

$$E \left[ (A_{it}^+)^{\zeta} \right] \leq E \left[ (A_{it} + H_{it})^{\zeta} \right]$$

Moreover, we have the upper bound:

$$E \left[ (A_{it} + H_{it})^{\zeta} \right] \leq \max(1, 2^{\zeta-1}) \left( E[A_{it}^{\zeta}] + E[H_{it}^{\zeta}] \right)$$

This inequality obtains because, for $\zeta > 1$, $x \to x^\zeta$ is convex, and for $\zeta \in [0, 1]$, $x \to x^\zeta$ is sub-additive.

Joining all these inequalities, and dividing by $E[A_{it} + H_{it}]^\zeta$, we obtain:

$$\frac{E[A_{it}^\zeta]}{E[A_{it} + H_{it}]^\zeta} \leq m_{it}^{\zeta} \leq \max(1, 2^{\zeta-1}) \frac{E[A_{it}^{\zeta}] + E[H_{it}^{\zeta}]}{E[A_{it} + H_{it}]^\zeta}$$

Finally, denote $p_{Dt}$ denotes the price-dividend ratio of the representative firm and $p_{Ht}$ the total value of human capital in the economy divided by the share of endowment distributed as labor income. We have

$$E[A_{it} + H_{it}]^\zeta/Y_{it}^\zeta = p_{it}^\zeta$$

$$E[A_{it}]^\zeta/Y_{it}^\zeta = p_{Dt}^\zeta \omega^\zeta$$

$$E[H_{it}]^\zeta/Y_{it}^\zeta = \omega^\zeta p_{Ht}^\zeta E[\chi_{it}^\zeta]$$

Therefore the $\zeta$ moment of total wealth converges a.s. to infinity iff the $\zeta$ moment of financial wealth converges to infinity a.s., which means that both distribution have the same tail index. \hfill \Box

**Proof of Proposition 6.** Applying Ito’s lemma on Proposition 2, we get

$$d \ln x_t = \left( \mu_{w,At} - \frac{1}{2} \sigma_{w,At}^2 + (\delta + n) \left( \frac{\pi_A}{x_t} \phi_t - 1 \right) \right) dt + \sigma_{w,At} dZ_t \tag{A49}$$

Since $\ln(x_t)$ is ergodic, its drift must average to zero. This gives:

$$0 = E \left[ \mu_{w,At} - \frac{1}{2} \sigma_{w,At}^2 + (\delta + n) \left( \frac{\pi_A}{x_t} \phi_t - 1 \right) \right] \tag{A50}$$

This expression allows me to replace $E \left[ \mu_{w,At} - \frac{1}{2} \sigma_{w,At}^2 \right]$ in the expression for the tail index. \hfill \Box

**Proof of Proposition 7.** This proof adapts Gomez (2018) to the case of this model. Applying Ito’s lemma on Equation (49) gives the law of motion of the quantile $q_t$

$$0 = -g_t(q_t) \frac{dq_t}{dt} + \int_{q_t}^{+\infty} \frac{d\tilde{q}_t(w)}{d\tilde{q}_t} d\tilde{q}_t - \sigma [d\tilde{q}_t(q_t)] \sigma [dq_t] \tag{A51}$$

57
where \( \sigma(dg_t(q_t)) \) and \( \sigma[dq_t] \) denote respectively the volatility of \( g_t(q_t) \) and \( q_t \). Applying Ito’s lemma on Equation (50) gives the law of motion of the top share \( S_t \):

\[
dS_t = -q_t g_t(q_t) dq_t + \int_{q_t}^{\infty} w dg_t(w) dw - q_t \sigma[dg_t(q_t)] \sigma[dq_t] dt - \frac{1}{2} g_t(q_t) \sigma[dq_t]^2 dt
\]  

(A52)

Using the law of motion for \( q_t \) from Equation (A51), we obtain the law of motion of \( S_t \):

\[
dS_t = \int_{q_t}^{\infty} (w - q_t) dg_t(w) dw - \frac{1}{2} g_t(q_t) \sigma[dq_t]^2 dt
\]

\[
= \int_{q_t}^{\infty} (w - q_t) dg_t(w) dw - \frac{1}{2} \left( \int_{q_t}^{\infty} \sigma[dg_t(w)] dw \right)^2 dt
\]  

(A53)

where

\[
dg_t = \pi_A dg_{At} + (1 - \pi_A) dg_{Bt}
\]  

(A54)

and the law of motion of \( g_{jt} \) for \( j \in \{A, B\} \) is given by the Kolmogorov-Forward equation from Lemma 3

\[
dg_{jt} = \left( -\mu_{w,t} \partial_w (wg_{jt}) + \frac{1}{2} \sigma_{w,t}^2 \partial_{ww} (w^2 g_{jt}) + (\delta + n)(g_{\chi,t} - g_{jt}) \right) dt - \sigma_{w,t} \partial_w (wg_{jt}) dZ_t
\]  

(A55)

Plugging it into (A53), we obtain

\[
dS_t = \int_{q_t}^{\infty} (w - q_t) \left( \sum_{j \in \{A, B\}} \pi_j \left( -\mu_{w,t} \partial_w (wg_{jt}) + \frac{1}{2} \sigma_{w,t}^2 \partial_{ww} (w^2 g_{jt}) \right) dt - \sigma_{w,t} \partial_w (wg_{jt}) dZ_t \right) dw
\]

\[+ \int_{q_t}^{\infty} (w - q_t) \left( \sum_{j \in \{A, B\}} \pi_j (\delta + n)(g_{\chi,t} - g_{jt}) \right) dt
\]

\[+ \frac{1}{2} \left( \sum_{j \in \{A, B\}} \pi_j \sigma_{w,t} \partial_w (wg_{jt}) \right)^2 dt dw
\]  

(A56)

Integrating by parts and rearranging:

\[
dS_t = \left( \int_{q_t}^{\infty} \sum_{j \in \{A, B\}} \mu_{w,t} \pi_j g_{jt}(w) \right) dt + \left( \int_{q_t}^{\infty} \sum_{j \in \{A, B\}} \sigma_{w,t} \pi_j g_{jt}(w) dw \right) dZ_t
\]

\[+ (\delta + n) dt \left( q_t c - S_t + \int_{q_t}^{\infty} (w - q_t) g_{\chi,t}(w) dw \right)
\]

\[+ \frac{g_t^2(q_t)}{2} dt \left( \sum_{j \in \{A, B\}} \frac{\sigma_{w,t}^2 \pi_j g_{jt}(q_t)}{g_t(q_t)} - \left( \sum_{j \in \{A, B\}} \frac{\pi_j g_{jt}(q_t)}{g_t(q_t)} \right)^2 \right)
\]  

(A57)

Dividing by \( S_t \), we obtain Proposition 7
Appendix D

D.1 The Exposure of Human Capital in Top Percentiles

The model gives sharp predictions for the wealth exposure of top percentiles, where wealth is defined using a concept that includes human capital. However, in the data, I only measure the exposure of financial wealth, which does not include human capital. In this section, I argue that, for households at the top of the wealth distribution, the difference between the exposure of total wealth and the exposure of financial wealth is quantitatively small.

Formally, for a given household in the economy, denote $a$ his financial wealth and $h$ his human capital. Denote $\omega = h/(a + h)$ the ratio of human capital wealth to total wealth. Following the log linearization in Campbell (1996), the return on total wealth can be written as a weighted average of the return of financial assets and the return of human capital:

$$\log \frac{a_{t+1} + h_{t+1}}{a_t + h_t} \approx (1 - \omega) \log \frac{a_{t+1}}{a_t} + \omega \log \frac{h_{t+1}}{h_t}$$

(A58)

Projecting this approximation on stock returns, the exposure of total wealth to stock market returns, $\beta_{a+h}$, can be written as a weighted average of the exposure of financial wealth, $\beta_a$, and the exposure of human capital, $\beta_h$:

$$\beta_{a+h} \approx (1 - \omega)\beta_a + \omega\beta_h$$

(A59)

This allows me to express the relative bias between the exposure of total wealth and the exposure of financial wealth:

$$\frac{\beta_{a+h} - \beta_a}{\beta_a} = \omega \left( \frac{\beta_h - \beta_a}{\beta_a} \right)$$

(A60)

The bias increases with the ratio of human capital to total wealth, $\omega$, and with the relative difference between the exposure of financial wealth and of human capital $(\beta_h - \beta_a)/\beta_a$.

The main observations is that this bias is small as long as $\omega$, the share of total wealth in human capital, is small. This appears to be the case for households at the very top of the wealth distribution. Indeed, the IRS reports that labor income represents 8.5% of total income for top households in the U.S.\footnote{See https://www.irs.gov/pub/irs-soi/13intop400.pdf.}. Assuming the same capitalization rate for human capital and financial assets, this suggests that human capital represents less than one tenth of financial wealth for households at the top of the wealth distribution.\footnote{This result is linked to the fact that the distribution of financial wealth has a thinner tail than the distribution of financial wealth, see Toda (2012).}

With $\omega \approx 10\%$, an exposure of human capital is between 0 and 2 only bias the estimate by $-10\%$ or $10\%$ (in relative term).

As a supplementary check, I regress labor income growth of the top 0.01% on excess stock market returns. This captures the exposure of human capital if the discount rate associated with human capital is...
constant over time. Table A2 reports an elasticity of 0.37 for the top 0.01% of households. After joining the estimates for \( \omega \) and \( \beta_h \), I conclude that the bias represents \( 0.1 \times (0.37/0.95 - 1) = -6\% \) of the estimated exposure \( \beta \).

D.2 Calibrated Parameters

I calibrate the human capital using the life cycle evolution of labor income, following Gârleanu and Panageas (2015). The life cycle income of households \( G(u) \) is a sum of two exponentials approximating the hump shaped pattern of earnings observed in the data:

\[
G(u) = B_1 e^{-\delta_1 u} + B_2 e^{-\delta_2 u}
\]

(A61)

with \( B_1 = 30.72, B_2 = -30.29 \).

I chose the share of endowment distributed as labor income as \( \omega = 92\% \), following Gârleanu and Panageas (2015).55

Appendix E Resolving the Tension

E.1 Transitions

Proof of Proposition 8. Denote \( m_{At}^\xi \), the \( \xi \)-moment of relative wealth for households in group \( A \), i.e.

\[
m_{At}^\xi = \left( \int_{i \in I_{At}} w_{it}^\xi dt \right) / N_t
\]

\[
= \pi_A \int_{-\infty}^t (\delta + \eta)e^{-(\delta + \eta + \tau)(t-s)} w_{Ats}^\xi ds
\]

Applying Ito’s lemma

\[
dm_{At}^\xi = \left( \xi \mu_{w,At} + \frac{\xi(\xi - 1)}{2} \sigma_{w,At}^2 - \delta - n - \tau \right) m_{At}^\xi dt + (\delta + n)\pi_A m_{At}^\xi dt + \xi \sigma_{w,At} m_{At}^\xi dZ_t
\]

(A62)

Using Lemma 3, the distribution of wealth of households in group \( A \) has a tail index given by \( \zeta \).

Similarly, denote \( m_{Bt}^\xi \) the \( \xi \)-moment of wealth for households in group \( B \)

\[
m_{Bt}^\xi = \left( \int_{i \in I_{Bt}} w_{it}^\xi di \right) / N_t
\]

\[54\]See Parker and Vissing-Jorgensen (2009) or Guvenen et al. (2017) for similar measures of the elasticity of top income shares to aggregate shocks. As in these papers, I also find that the elasticity is the labor income to aggregate shocks increased after 1980.

\[55\]A lower value for the labor share would only strengthen my result: since a lower labor share decreases the wealth of arriving agents, it rises the average wealth growth of existing households. In equilibrium, this tends to increase the interest rate. But the model already tends to give an interest rate that is too high, as seen in Table 5.
We have, using Ito’s lemma

\[ dm^\xi_{Bt} = \left( \xi \mu_{w_{Bt}} + \frac{\xi(\xi - 1)}{2} \sigma_{w_{Bt}}^2 - \delta - n \right) m^\xi_{Bt} dt + \left( \delta + n \right) (1 - \pi_A) m^\xi_{A_t} dt + \tau m^\xi_{A_t} dt + \xi \sigma_{w_{Bt}} m^\xi_{Bt} dZ_t \]

The distribution of wealth for households in group \( B \) inherits the tail \( \zeta \) due to the transition from group \( A \) to group \( B \).

Solving the Model  To solve the model, I follow the same steps as Appendix B.2. The wealth evolution Proposition 1 is not modified, because agents do not forecast any change in preferences.

E.2 Time-Varying Investment Opportunities

Solving the Model  To solve the augmented model, I look for a Markov equilibrium with two state variables \((x_t, \nu_t)\). The law of motion for \( x_t \) is the same as Proposition 2 while the law of motion for \( \nu_t \) is exogenously given by (60). Given the law of motion for \( x_t, \nu_t \), one can express the drift and the volatility of all processes through Ito’s lemma and proceed similarly to the baseline model.
Table A1: The Equity Share Increases Across the Wealth Distribution

<table>
<thead>
<tr>
<th>Groups of Households Defined by Wealth Percentiles</th>
<th>All Households</th>
<th>1% – 0.1%</th>
<th>0.1% – 0.01%</th>
<th>Top 0.01%</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A: All Sample</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Equity Share</td>
<td>40.8%</td>
<td>55.8%</td>
<td>65.8%</td>
<td>73.9%</td>
</tr>
<tr>
<td>Public Equity</td>
<td>20.2%</td>
<td>22.0%</td>
<td>21.1%</td>
<td>19.6%</td>
</tr>
<tr>
<td>Private Equity</td>
<td>20.6%</td>
<td>33.9%</td>
<td>44.6%</td>
<td>54.4%</td>
</tr>
<tr>
<td>Non Actively Managed</td>
<td>2.4%</td>
<td>4.5%</td>
<td>6.3%</td>
<td>7.8%</td>
</tr>
<tr>
<td>Actively Managed</td>
<td>18.2%</td>
<td>29.4%</td>
<td>38.4%</td>
<td>46.6%</td>
</tr>
<tr>
<td><strong>Panel B: Stockholders</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Is Stockholder</td>
<td>45.9%</td>
<td>90.7%</td>
<td>91.2%</td>
<td>91.0%</td>
</tr>
<tr>
<td>Equity Share among Stockholders</td>
<td>44.7%</td>
<td>56.0%</td>
<td>65.9%</td>
<td>76.0%</td>
</tr>
<tr>
<td><strong>Panel C: Entrepreneurs</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Is Entrepreneur</td>
<td>10.5%</td>
<td>62.1%</td>
<td>69.8%</td>
<td>78.5%</td>
</tr>
<tr>
<td>Equity Share among non-Entrepreneurs</td>
<td>26.8%</td>
<td>40.7%</td>
<td>50.0%</td>
<td>57.9%</td>
</tr>
<tr>
<td><strong>Panel D: Stock Options Holders</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Received Stock Options</td>
<td>6.4%</td>
<td>11.2%</td>
<td>11.5%</td>
<td>6.1%</td>
</tr>
<tr>
<td>Equity Share among non Stock Options Holders</td>
<td>44.7%</td>
<td>56.0%</td>
<td>65.9%</td>
<td>76.0%</td>
</tr>
<tr>
<td>Share of Total Wealth</td>
<td>20.7%</td>
<td>7.7%</td>
<td>3.8%</td>
<td></td>
</tr>
<tr>
<td>Labor Income / Wealth</td>
<td>12.6%</td>
<td>2.9%</td>
<td>1.6%</td>
<td>0.7%</td>
</tr>
</tbody>
</table>

Notes: Data from SCF 1989-2013. The variable Equity Share is defined as private equity + public equity over networth: 
(equity + bus) / networth. Stockholders are defined as the households that hold public equity. Entrepreneurs are defined as the households with an active management role in one of the company they invest in.
Table A2: The Exposure of Labor Income Growth Across the Wealth Distribution:

<table>
<thead>
<tr>
<th>Total Income Group of Households Defined by Wealth Percentiles</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Excess Stock Returns</td>
<td>0.25</td>
<td>0.14</td>
<td>0.17</td>
<td>0.37*</td>
</tr>
<tr>
<td>N</td>
<td>92</td>
<td>92</td>
<td>92</td>
<td>92</td>
</tr>
</tbody>
</table>

Notes. The table reports the results of the regression of the growth of labor income on asset returns.

\[
\log \frac{Y_{G,t-1+h}}{Y_{G,t-1}} - h \log R_{ft} = \alpha_{Gh} + \beta_{Gh}(\log R_{M} - \log R_{f}^t) + \epsilon_{Ght}
\]

The total labor income received by within a top percentile is obtained from Saez and Zucman (2016).

Estimation is via OLS. Standard errors in parentheses and estimated using Newey-West with 4 lags. *, **, *** indicate significance at the 0.1, 0.05, 0.01 levels, respectively.

Table A3: The Exposure to Stock Returns Across the Wealth Distribution:

**Saez and Zucman (2016) Series**

<table>
<thead>
<tr>
<th>Wealth Growth Within Percentile Thresholds</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Excess Stock Returns</td>
<td>0.49***</td>
<td>0.49***</td>
<td>0.66***</td>
</tr>
<tr>
<td>N</td>
<td>96</td>
<td>96</td>
<td>96</td>
</tr>
</tbody>
</table>

Notes: The table reports the results of the regression of the growth of the wealth growth of households in a given percentile group on asset returns.

Estimation is via OLS. Standard errors in parentheses and estimated using Newey-West with 4 lags. *, **, *** indicate significance at the 0.1, 0.05, 0.01 levels, respectively.
Figure A1: The Equity Share Increases Across the Wealth Distribution

(a) Top Percentiles Linearly Spaced

(b) Top Percentiles Log-linearly Spaced

**Notes.** Figure A1a plots the average equity share within 20 linearly spaced percentile bins in the wealth distribution. Figure A1b plots the average equity share within 20 logarithmically spaced percentile bins in the wealth distribution. The horizontal line represents the average equity share. The vertical line splits the set of households in two: households on either side of the vertical line own half of total wealth (this corresponds to top percentile $\approx 3\%$). Data from the Survey of Consumer Finance (SCF), a cross-sectional survey of US households from 1989 to 2013. The equity share is constructed as $(\text{equity} + \text{bus}) / \text{networth}$. 
References


