Asset Prices and Wealth Inequality*

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Abstract

I examine recently available data on the top of the wealth distribution through the lens of asset pricing models with heterogeneous agents. A standard model where agents have heterogeneous preferences matches three key facts about the relation between asset prices and wealth inequality: (1) the wealth distribution is fat-tailed, due to the high wealth growth of households in top percentiles (2) when stock market returns are high, wealth inequality increases (3) higher wealth inequality predicts lower future excess stock returns. Quantitatively, however, a standard model that matches the wealth distribution cannot fully account for the volatility of asset prices in equilibrium: to match the high volatility of asset prices, the model would require such a large degree of preference heterogeneity that it would give rise to a wealth distribution close to Zipf’s law, i.e. with a right tail much thicker than the data.

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1 Introduction

Recent empirical studies have documented important fluctuations in wealth inequality over the past century.\(^1\) Volatile stock market returns potentially account for these fluctuations. Conversely, a large theoretical literature in asset pricing examines the role of household heterogeneity in shaping asset prices, but seldom considers its implication on wealth inequality. In this paper, I use this recently available data on wealth inequality to examine empirically and theoretically the relationship between asset prices and the wealth distribution.

I focus on the following mechanism. Risk-tolerant investors hold more risky assets, accumulate more wealth, and disproportionately end up at the top of the wealth distribution. As a consequence, in periods when stocks enjoy large realized returns, investors at the top of the wealth distribution gain more than the rest, i.e. wealth inequality increases. In turn, as a larger share of wealth is owned by risk-tolerant households, aggregate demand for risk increases, which lowers risk premia and pushes up asset prices, i.e. higher wealth inequality predicts lower future excess-returns. I confirm empirically this joint dynamic between asset prices and wealth inequality. In response to a realized stock return of 10%, the wealth share of the top 0.01% increases by 4.5%. In turn, a 10% increase in the wealth share of the top 0.01% predicts lower future excess-returns by one percentage point.

I then evaluate whether this mechanism can quantitatively account for asset prices and the wealth distribution in equilibrium. I use the reduced-form evidence I documented earlier to estimate a state-of-the-art asset pricing model with heterogeneous agents. I find that a model consistent with the wealth distribution cannot fully account for the volatility of asset prices. Specifically, the model cannot generate the high volatility of asset prices without implying an excessive level of inequality compared to the data.

The paper proceeds in three stages. First, I present two stylized facts on the relationship between asset prices and the wealth distribution. I first show that, because top households tend to hold more equity, stock market returns generate large fluctuations in wealth inequality. To show this, I use the series of top wealth shares constructed from tax filings by Kopczuk and Saez (2004) and from Forbes 400 to estimate the exposure of the top percentiles to stock market returns. In response to a realized stock return of 10%, the average wealth increases by 5%, while the average wealth for the top 0.01% increases by 9.5%. Therefore, the wealth share of the top 0.01% increases by the difference of 4.5%.

\(^1\)See, for instance, Kopczuk and Saez (2004), Piketty (2014), and Saez and Zucman (2016).
The flip side of this relationship is that, in an economy where inequality is high, the share of wealth owned by risk-tolerant investors is high, and therefore, in equilibrium, risk premia are low. Thus, higher inequality should predict lower future returns. Indeed, in the data, I find that a 10% increase in the wealth share of the top 0.01% predicts lower future excess-returns by one percentage points over the following year.

Second, I examine whether those facts through the lens of a state-of-the-art asset pricing model with heterogeneous agents. Specifically, I study the joint dynamics of asset prices and wealth inequality in a continuous-time, overlapping generations framework where agents differ with respect to their relative risk aversion (RRA) and elasticity of intertemporal substitution (EIS), following Gârleanu and Panageas (2015). The model can qualitatively generate my two stylized facts. Using an analytical version of Campbell-Shiller decomposition, the effect of wealth inequality on asset prices can be decomposed into an “excess-returns channel”, that depends on the sensitivity of expected excess-returns to wealth inequality (which is mostly driven by the heterogeneity in RRA), and a “risk-free rate channel”, that depends on the sensitivity of the risk-free to wealth inequality (which is mostly driven by the heterogeneity in EIS).

One key contribution of my analysis is to study the wealth distribution implied by the model. Because the wealth dynamics of individual households depends on the aggregate state of the economy, the wealth distribution fluctuates over time. In this dynamic environment, I show that the wealth distribution exhibits a thick right tail. I offer an analytical characterization for its tail exponent: it is determined by the average logarithmic wealth growth rate of top households relative to the economy.

Third, to assess whether the model is quantitatively consistent with the reduced form evidence, I estimate the model on moments related to asset prices and to the wealth distribution. I show that a model consistent with the wealth distribution cannot fully match the high volatility of asset prices. Intuitively, to match the volatility of asset prices with small aggregate shocks, the model requires a large degree of heterogeneity between agents. In the long run, this degree of heterogeneity generates a wealth distributions close to Zipf’s law, whereas the wealth distribution has a much thinner tail in the data. This tension arises independently of the source of preference heterogeneity: it is present whether households differ with respect to their relative risk aversion, intertemporal elasticity of substitution, or/and subjective discount rates.

Overall, these results suggest a strong link between asset prices and wealth inequality. While a number of studies focus on the role of the risk-free rate of return in shaping the wealth distribution,\(^2\)

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\(^2\)See, for instance, Piketty (2014) and Acemoglu and Robinson (2015).
I document a more important role for the rate of return on risky assets: a model where agents have different risk aversion explains well both the level and the dynamic of wealth inequality over time. After estimating a model on moments about the wealth distribution, however, I show that there is not enough heterogeneity between households to fully explain the high volatility of asset prices in equilibrium.

**Related Literature.** A large literature in household finance examines the heterogeneity in portfolio choice across the wealth distribution. The fact that the share of wealth invested in risky assets increases with wealth is documented in Guiso et al. (1996), Carroll (2000), Campbell (2006), Wachter and Yogo (2010), Roussanov (2010), Calvet and Sodini (2014), and Fagereng et al. (2016). As in Bach et al. (2015) which studies Swedish households, I stress the heterogeneous exposure to aggregate risk across the wealth distribution. Compared to these papers, I focus on the role of this portfolio heterogeneity for the dynamics of wealth inequality. In particular, I am interested in the elasticity of top wealth shares to stock market returns. This empirical strategy also relates this paper to Parker and Vissing-Jørgensen (2009), which measures the exposure of the labor income of top households to aggregate income shocks.

This paper contributes to the growing literature on wealth inequality. On the empirical side, I rely critically on the recent wealth shares constructed by Kopczuk and Saez (2004). On the theoretical side, random growth theories of the wealth distribution include Wold and Whittle (1957), and, more recently, Benhabib et al. (2011), Benhabib et al. (2015b), Benhabib et al. (2016), Jones (2015), Acemoglu and Robinson (2015), and Cao and Luo (2016). While these papers focus on static economies, I examine the wealth distribution in an economy that evolves stochastically. Even though the wealth distribution evolves over time, I offer an analytical characterization of its right tail, extending the tools developed by Luttmer (2012) and Gabaix et al. (2016). I show that, in presence of aggregate shocks, the tail exponent of the wealth distribution is determined by the average logarithmic wealth growth rate of top households relative to the economy. The role of this object in shaping the wealth distribution ties this paper to the literature on growth-optimal portfolios.3

This literature of asset pricing models with heterogeneous agents, initiated by Dumas (1989), is too large to summarize here. Most relevant to this paper, Gărleanu and Panageas (2015) presents an overlapping generation model where agents have heterogeneous preferences that can match asset pricing models.4

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3This literature is initiated by Kelly (1956). See also see Blume et al. (1992), or, more recently, Borovička (Forthcoming).
prices. My contributions compared to this paper is to study analytically the behavior of the wealth distribution in the model and to test the model using the recently available data on the top of the wealth distribution. I highlight a key tension in these models between matching asset prices and the wealth distribution. While the OLG framework allows the wealth distribution to be stationary in presence of preference heterogeneity, its right tail tends to be too thick compared to the data.

A growing literature studies the impact of investment heterogeneity on wealth inequality and asset prices. Gollier (2001) is an early example that examines theoretically the importance of the wealth distribution for asset prices. Barczyk and Kredler (2016) examines theoretically the role of inequality and incomplete markets on asset prices. Eisfeldt et al. (2016) examines the joint relation between the wealth distribution and asset prices across markets with different expertises. Kacperczyk et al. (2018) studies the role of investor sophistication for the recent rise in capital income inequality. Empirically, Favilukis (2013) examines the role of changes in participation cost and wage inequality on asset prices. Johnson (2012) examines the role of income inequality shocks for the cross-section of returns. Most relevant to this paper, Toda and Walsh (2016) independently shows that fluctuations in income inequality negatively predict future excess stock returns, using the series on top income shares from Piketty and Saez (2003). My paper confirms these results using the series of top wealth shares shares from Kopczuk and Saez (2004). Relative to this paper, I examine the magnitude of this effect through a quantitative model. This leads me to highlight some other moments, more precisely estimated, that discipline the effect wealth inequality can have on asset prices (e.g. the elasticity of top wealth shares to stock market returns and the tail exponent of the wealth distribution).

Road Map The rest of my paper is organized as follows. In Section 2, I document two key stylized facts about the relation between wealth inequality and asset prices. In Section 3, I present a standard asset pricing model with heterogeneous agents to interpret these findings. In Section 4, I characterize analytically the wealth distribution in the model. In Section 5, I highlight the key tension in the model between matching jointly asset prices and the wealth distribution. Section 6 concludes.

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This paper focuses on the role of aggregate shocks for the wealth distribution. Other papers consider the role of displacement shocks (Gărleanu et al. (2012), Gărleanu and Panageas (2017)), idiosyncratic shocks (Constantinides and Duffie (1996), Storesletten et al. (2007), Kogan et al. (Forthcoming), Schmidt (2016)), fluctuating capital shares (Lettau et al. (2016), Greenwald et al. (2014)) or fluctuating tax rates (Pastor and Veronesi (2016)).


2 Data and Facts

I now analyze data about the top of the wealth distribution to document two stylized facts predicted by heterogeneous agents models. In particular, I focus on the following mechanism. Risk-tolerant households invest more in risky assets and disproportionately end up at the top of the wealth distribution. In periods when stocks enjoy large realized returns, investors at the top of the wealth distribution gain more than the rest; thus, inequality increases. In turn, as a larger share of wealth falls into the hands of risk-tolerant households, the aggregate demand for risk increases, which lowers risk premia; thus, higher inequality predicts lower future returns. After introducing the data, I document facts reflecting each step of this mechanism.

2.1 Data

Wealth Shares I am interested in measuring changes in the wealth distribution and their relationship to stock returns. Therefore, I need yearly estimates of the wealth distribution that cover several business cycles. I use two datasets that, together, cover most of the last 100 years.

The first wealth series is the annual series of top wealth shares constructed by Kopczuk and Saez (2004). This series is constructed from estate tax returns, which report the wealth of deceased individuals, above a certain wealth threshold. From the wealth distribution of the deceased, Kopczuk and Saez (2004) estimates the wealth distribution of the living using the mortality multiplier technique, which amounts to re-weighting each estate tax return by the inverse probability of death (depending on age and gender). The series is constructed using the universe of estate tax returns during the 1916-1945 period, and a stratified sample of micro-files for 1965, 1969, 1972, 1975 and 1982-2000. Total wealth uses the household balance sheets of the US Financial Accounts after 1945. Before 1945, Kopczuk and Saez (2004) reports estimates based on the same concepts and methods as the Financial Accounts, following Wolff and Marley (1989).

Another data series about wealth inequality is Saez and Zucman (2016). They construct top wealth shares from income tax returns, using a capitalization method. However, the series builds in smoothing over time, which makes it harder to examine the joint dynamics of asset prices and top wealth shares. I compare the two series more thoroughly in Appendix A.

I supplement the series of top wealth shares with the list of the wealthiest 400 Americans constructed by Forbes Magazine every year since 1982, which offers an unparalleled view on the right tail of the wealth distribution. The list is created by a dedicated staff of the magazine, based
on a mix of public and private information.\textsuperscript{5} The total wealth of individuals on the list accounts for approximately 1.5\% of total aggregate wealth in 2010.\textsuperscript{6}

**Asset prices**  For asset prices, I use yearly stock market returns and risk-free rates from Shiller (2015).\textsuperscript{7} The excess stock market return is measured as the log stock market return minus the log risk-free rate. Data on the price-dividend ratio and the price-payout ratio comes from Welch and Goyal (2008).

## 2.2 Wealth Exposure to the Stock Market Across the Wealth Distribution

The basic building block of heterogeneous agents models is that there is a group of investors that is disproportionately exposed to aggregate shock. Following a positive return, these households gain more relative to other households; therefore, the wealth distribution fluctuates.

To measure the heterogeneity in risk exposure across the wealth distribution, I estimate the wealth exposure to stock market returns at different percentiles of the distribution. More precisely, I define the exposure of households in a given group as the slope estimate of a regression of the growth of total wealth in the group on excess stock market returns, i.e.,

\[
\log \left( \frac{W_{G,t-1+h}}{W_{G,t-1}} \right) - h \log R_{ft} = \alpha_{Gh} + \beta_{Gh}(\log R_{Mt} - \log R_{ft}) + \epsilon_{Ght} \tag{1}
\]

where \(W_{G,t}\) denotes the total wealth of households in the percentile group \(G\) in year \(t\), \(\log R_{Mt}\) denotes the log stock market return, and \(\log R_{ft}\) denotes the log risk-free rate. The dependent variable is the log ratio of wealth in the percentile group \(G\) from year \(t - 1\) to year \(t - 1 + h\). Note that the starting year is \(t - 1\): this is because \(W_{Gt}\) represents the average wealth owned by the group during year \(t\), rather than the wealth owned at the beginning of the year.

The first four columns in Table 1 (Panel A) report the estimates for \(\beta_{G4}\), the wealth exposure to the stock market at the 4 year horizon \((h = 4)\), for four groups of households: all households, households in the top 1\% to 0.1\%, households in the top 0.1\% to 0.01\%, and households in the top 0.01\%. The estimated exposure \(\beta_{G4}\) increases with the top percentiles, from 0.48 for the average

\textsuperscript{5}Forbes Magazine reports: “We pored over hundreds of Securities Exchange Commission documents, court records, probate records, federal financial disclosures and Web and print stories. We took into account all assets: stakes in public and private companies, real estate, art, yachts, planes, ranches, vineyards, jewelry, car collections and more. We also factored in debt. Of course, we don’t pretend to know what is listed on each billionaire’s private balance sheet, although some candidates do provide paperwork to that effect.”

\textsuperscript{6}Recent empirical studies examining the Forbes 400 list also include Klass et al. (2006) and Kaplan and Rauh (2013).

household, to 0.95 for households in the top 0.01%. The last column of Panel A in Table 1 reports the wealth exposure of the Top 400 from Forbes. The estimates for households in the extreme tail of the distribution are similar in magnitude to the estimates for households in the top 0.01% from tax data. This is reassuring, because these datasets are constructed from two completely different sources. In short, top households are twice as exposed to stock market returns as the representative household.

Since top households are comparatively more exposed to the stock market, high stock market returns increase inequality. Panel B of Table 1 confirms this relationship by regressing top wealth shares on stock market returns. The estimate 0.48 corresponds to the difference of exposure between households at the top and the average household ($\approx 0.95 - 0.48$). Moreover, the regression shows that the difference is statistically significant.

**Horizon** Top wealth shares, as measured by Kopczuk and Saez (2004) and Forbes, may not react immediately to changes in stock market returns. This is because a large share of wealth in top percentiles is held in privately-held assets, and the value of these assets tends to lag the price movements of public benchmarks.\(^8\) This introduces a stale-pricing problem: top wealth shares may not react immediately to changes in stock market returns.

To handle this stale-pricing problem, I examine the reaction of top wealth shares at different horizons. Figure 1 plots the estimate of $\beta_h$ obtained by making the horizon $h$ varies from $h = 1$ to $h = 8$ in Equation (1). This traces the impulse response of top wealth shares to a stock market return shock.\(^9\) The plot shows that the estimates tend to increase from $h = 1$ to $h = 4$, consistent with the stale pricing problem. The estimate peaks at the four year horizon, justifying the choice of $h = 4$ in the regressions above.

**Composition** Top percentiles do not necessarily include the same individuals over time - some people enter and drop from the top every year. This may bias the regressions above: if there were more entrants in top percentiles in times of high stock market returns, the exposure of top wealth shares to stock market returns would overestimate the actual wealth exposure of top households. To address this concern, I use the panel dimension of Forbes 400. Each year, I construct the average wealth growth of households in the Top 400, whether or not they drop from the list by the end of the year. This yearly series differs from the total growth of the top 400, by removing the

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\(^8\)See in particular Getmansky et al. (2004). Relatedly, Brav et al. (2002), Malloy et al. (2009) show that reported consumption growth for richer households is more correlated to aggregate consumption growth at longer horizon.

effect of compositional changes on the growth of top wealth shares.\textsuperscript{10} Table 2 shows that I obtain very similar results with this new series: compositional changes play no role in driving the stock market exposure of top percentiles. This comes from the fact that fluctuations in the idiosyncratic variance of wealth growth (which drives the entry and exit of households in top percentiles) tends to be much smaller than fluctuations in stock market returns.

2.3 Top Wealth Shares and Future Excess-Returns

The previous evidence suggests that wealthy households are more willing to take on aggregate risk. The flip side of this relationship is that, as top wealth shares increase, wealth is rebalanced from risk-averse households to risk-tolerant households; therefore, the total demand for risk in the economy increases. In equilibrium, the compensation for holding risk decreases. Hence, higher top wealth shares should predict lower future excess-returns.

I measure the predictive power of top wealth shares by regressing excess stock returns on the wealth share of the top 0.01%:

\[
\sum_{1 \leq h \leq H} \log R_{M,t+h} - \log R_{f,t+h} = \alpha + \beta_H \log \text{Wealth Share Top 0.01\%}_t + \epsilon_H t
\]

where \(h\) denotes the horizon, \(\log R_{M,t}\) denotes the log stock market return, and \(\log R_{f,t}\) denotes the log risk-free rate.

The first line in Table 3 reports the results of the predictability regression at the one-year and three-year horizons using the wealth share of the top 0.01% as a predictor. The estimate is negative and significant at the 10% level. Quantitatively, a 10% increase in the wealth share of the top 0.01% is associated with a decrease of excess-returns by one percentage point over the next year.

Figure 2 plots the wealth share of the top 0.01% along with a moving average of excess stock returns over the following eight years. Fluctuations in the wealth share of the top 0.01% do a particularly good job at tracking the low-frequency fluctuations in excess stock returns. In particular, excess stock returns were low when inequality was high in the 1920s. Excess stock returns increased following the decrease in inequality in the 1930s, and decreased following the increase in inequality in the 1980s.

It is well known that, for a predictor that is is persistent and correlated with returns, like top wealth shares, conventional t-statistics are misleading.\textsuperscript{11} To address this concern, I rely on a test developed in Campbell and Yogo (2006), which is valid even when the predictor variable has a root

\textsuperscript{10} See Gomez (2018) for more details about this construction.

\textsuperscript{11} See, for instance, Elliott and Stock (1994) and Stambaugh (1999).
close to or larger than one.\footnote{The test can only be done for the restricted sample without gaps in the predictor, i.e. 1917-1951.} As reported in Table 4, the hypothesis that the top wealth share has a unit-root cannot be rejected. Still, even after allowing for explosive dynamics in top wealth shares, the wealth share of the top 0.01% is found to significantly predict returns.

Another way to correct for the persistence of the predictor is to use a detrended version of the predictor (see Hodrick (1992)). Table 3 also reports the predictability regression using both the five-year difference and the detrended version of the top wealth share. The predictive power of top wealth shares remains significant.

Finally, I examine whether the information in the wealth share of the top 0.01% is subsumed by the price-dividend ratio, by running the following bivariate predictive regression:

\[
\sum_{1 \leq h \leq H} \log R_{M,t+h} - \log R_{f,t+h} = \alpha + \beta_H \log \text{Wealth Share Top 0.01\%}_t + \gamma_H dp_t + \epsilon_{Ht} \tag{3}
\]

Table 3 reports that the predictive power of top wealth shares remains substantial after one adds the dividend price or the dividend payout as a predictor.

I have shown that fluctuations in stock prices generate fluctuations in inequality, and, in turn, that the level of inequality determines future excess-returns. Those facts are at the heart of asset pricing models with heterogeneous agents. I now examine these facts within a quantitative model.

\section{Asset Pricing Model with Heterogeneous Preferences}

I consider a continuous-time pure-exchange economy. I present a model where overlapping generations of households differ in their preferences, which purposefully follows Gârleanu and Panageas (2015). I derive the dynamics of individual wealth, as well as the dynamics of the asset prices in the model.

\subsection{Setup}

**Demographics** The specification of demographics follows Blanchard (1985). Each agent faces a constant hazard rate of death $\delta > 0$. Total population size, denoted, $N_t$ grows at rate $\eta$. Therefore, during a short time period $dt$, a mass $\delta dt$ of the population dies and a new cohort of mass $(\delta + \eta)dt$ is born.

Moreover, there is a perfectly competitive annuity market: an agent $i$ with wealth $W_{it}$ receives $\delta W_{it}dt$ during the period $dt$ in exchange for promising the totality of their wealth if they die.
**Endowment**  I consider a continuous-time pure exchange economy. The aggregate endowment per capita exhibits i.i.d. growth, i.e. its law of motion is

\[
\frac{dY_t}{Y_t} = \mu dt + \sigma dZ_t \tag{4}
\]

where \( Z = \{Z_t \in \mathbb{R}|\mathcal{F}_t, t \geq 0\} \) is a standard Brownian motion defined on a probability space \((\Omega, P, \mathcal{F})\), equipped with a filtration \( \mathcal{F} = \{\mathcal{F}_t, t \geq 0\} \) with the usual conditions.

Markets are dynamically complete. There is a unique stochastic discount factor: denote \( r_t \) the risk-free rate and \( \kappa_t \) the price of risk.

**Preferences**  Agents have recursive preferences as defined by Duffie and Epstein (1992). They are the continuous-time versions of the recursive preferences of Epstein and Zin (1989). For an agent \( i \) with a consumption process \( C_i = \{C_{it} : t \geq 0\} \), their utility \( U_i = \{U_{it} : t \geq 0\} \) is defined recursively by:

\[
U_{it} = E_t \int_t^{+\infty} f_i(C_{is}, U_{is}) ds \tag{5}
\]

\[
f_i(C, U) = \frac{1}{1 - \frac{1}{\psi}} \left( \frac{C^{1 - \frac{1}{\psi}} - (1 - \gamma)(\rho + \delta)U}{((1 - \gamma)U)^{\frac{\gamma - \frac{1}{\psi}}{1 - \gamma}}} \right) \tag{6}
\]

These preferences are characterized by three parameters: the subjective discount rate \( \rho \), the coefficient of relative risk aversion (RRA) \( \gamma \), and the elasticity of intertemporal substitution (EIS) \( \psi \).

There are two types of agents, labeled A and B, that can differ with respect to their relative risk aversion \( \gamma_j \), and their elasticity of intertemporal substitution is \( \psi_j \). I denote A the risk-tolerant agent, i.e. \( \gamma_A > \gamma_B \). In calibrations explored below, this agent will also be more willing to substitute consumption over time, i.e. \( \psi_A \geq \psi_B \). Finally, at every point in time a proportion \( \pi_A \) of newly born agents are of type A.

**Household Problem**  Denote \( W_{it} \) the wealth of agent \( i \) at time \( t \). The problem of households is as follows. Household \( i \) chooses a consumption process \( C_i = \{C_{it} : t \geq s_i\} \) and a consumption exposure \( \sigma_{W_{it}} = \{\sigma_{W_{it}} : t \geq s_i\} \) such that for \( t \geq s_i \):\(^{13}\)

\[
V_{it} = \max_{C_{it}, \sigma_{W_{it}}} U_{it}(C_{it}) \tag{7}
\]

\[
dW_{it} = (r_t + \delta)W_{it} + \kappa_t\sigma_{W_{it}}(W_{it} - C_{it}) dt + \sigma_{W_{it}}W_{it}dZ_t \tag{8}
\]

\(^{13}\)I assume standard square integrability and transversality conditions.
Agents are endowed at birth with some wealth, of the form $\chi_i \phi_t W_t$, where $\chi_i$ is an individual specific shock with mean one, $\phi_t$ corresponds to the average wealth of a newborn agent relative to the economy, and $W_t$ denotes the wealth per capita in the economy.\footnote{This wealth depends on the labor income endowed to each agent for the economy. More details about the labor income process is given in Appendix C.}

### 3.2 Law of Motion of Household Wealth

I first characterize the law of motion of households’ wealth. This is an important object to study because it determines the dynamics of the wealth distribution.

Households with the same preference parameters face the same trade-off, irrespective of their wealth or age, due to the homogeneity of the utility function and the constant death rate. In particular, the consumption rate $C_{it}/W_{it}$ and the wealth exposure $\sigma_{W_{it}}$ are the same for all agents in the same group $j \in \{A, B\}$.

Denote $p_{jt}$ the wealth-to-consumption ratio of an agent in group $j$. Conjecture that the process $p_{jt}$ follows a diffusion process:

$$\frac{dp_{jt}}{p_{jt}} = \mu_{p_{jt}} dt + \sigma_{p_{jt}} dZ_t$$  \hspace{1cm} (9)

**Proposition 1** (Law of Motion for Households Wealth). The wealth of households in group $j \in \{A, B\}$ follows the law of motion

$$\frac{dW_{jt}}{W_{jt}} = \mu_{W_{jt}} dt + \sigma_{W_{jt}} dZ_t$$  \hspace{1cm} (10)

where $\mu_{W_{jt}}$ and $\sigma_{W_{jt}}$ are given by

$$\sigma_{W_{jt}} = \frac{\kappa_t}{\gamma_j} + \frac{1}{\psi_j - 1} \sigma_{p_{jt}}$$  \hspace{1cm} (11)

$$\mu_{W_{jt}} = \psi_j (r_t - \rho) + \frac{1 + \psi_j}{2\psi_j} \kappa_t^2 + \frac{\psi_j - \psi_j}{\psi_j - 1} \kappa_t \sigma_{p_{jt}} + \frac{1}{2(\psi_j - 1)} \sigma_{p_{jt}}^2 + \mu_{p_{jt}}$$  \hspace{1cm} (12)

The geometric volatility of wealth $\sigma_{W_{jt}}$ is the sum of two terms. The first term, the myopic demand, equals the ratio of the market price of risk to the relative risk aversion $\gamma_j$. The lower the relative risk aversion $\gamma_j$, the higher the myopic demand. The second term, the intertemporal hedging demand $H_{jt}$, captures deviations from the mean-variance portfolio due to changes in investment opportunities. If expected returns are countercyclical, this term is positive as long as $\gamma_j > 1$.\footnote{This wealth depends on the labor income endowed to each agent for the economy. More details about the labor income process is given in Appendix C.}
The geometric of wealth $\mu W_{jt}$ is the sum of three terms. The first term is a standard term due to intertemporal substitution, determined by the EIS $\psi_j$ and the difference between the interest rate $r_t$ and the subjective discount rate $\rho$: $\psi_j(r_t - \rho)$. Note that it is similar to the term for an infinite horizon investor: the OLG setup does not change the law of motion of individual wealth in response to a given set of prices. While agents in an OLG economy have an increased discount rate $\rho + \delta$ due to the probability of death, they also face an increased effective interest rate due to annuities, $r + \delta$.

The second term corresponds to the effect of higher risk exposure on wealth growth. Agents with lower risk aversion invest disproportionately in risky assets. Due to the compensation for holding more risk, they earn, on average, higher returns. This affects their consumption rate, through a combination of an income and a substitution effect. As $\psi_j$ rises, the substitution effect becomes increasingly important, which magnifies the effect of risk aversion on total wealth growth. The third term $\Phi_{jt}$ captures changes in investment opportunities.

### 3.3 Prices

**Market Clearing** The market clearing for consumption is

$$\int_{i \in I_{At}} C_{it} di + \int_{i \in I_{Bt}} C_{it} di = Y_t N_t \quad (13)$$

Denoting $p_t$ the ratio of total wealth to total consumption, i.e.

$$p_t = \frac{\int_{i \in I_{At}} W_{it} di + \int_{i \in I_{Bt}} W_{it} di}{\int_{i \in I_{At}} C_{it} di + \int_{i \in I_{Bt}} C_{it} di} \quad (14)$$

Market clearing for consumption can be rewritten as

$$\int_{i \in I_{At}} W_{it} di + \int_{i \in I_{Bt}} W_{it} di = p_t Y_t N_t \quad (15)$$

**Markov Equilibrium** For the purpose of determining prices, we can abstract from the distribution of wealth within each group: we only need to keep track of the share of aggregate wealth that belongs to the agent in group $A$:

$$x_t = \frac{\int_{i \in I_{At}} W_{it} di}{\int_{i \in I_{At}} W_{it} di + \int_{i \in I_{Bt}} W_{it} di} \quad (16)$$

Because agents in group $A$ choose a different wealth exposure compared to agents in group $B$, the process $x_t$, is stochastic. The next proposition characterizes the law of motion of $x_t$. 13
Proposition 2. The law of motion of $x$ is

$$dx_t = \mu_{xt} dt + \sigma_{xt} dZ_t$$ (17)

where $\mu_{xt}$ and $\sigma_{xt}$ are given by

$$\sigma_{xt} = x_t(\sigma_{W_{At}} - \sigma - \sigma_{p_t})$$ (18)

$$\mu_{xt} = x_t(\mu_{W_{At}} - \mu - \mu_{p_t} - \sigma_{p_t}) - \sigma_{xt}(\sigma + \sigma_{p_t}) + (\delta + \eta)(\pi_{A\phi_t} - x_t)$$ (19)

The volatility of $x_t$ is given by the difference between the wealth volatility of agents in group $A$ and the volatility of aggregate wealth. The drift of $x_t$ is the sum of three terms. The first term is the difference between the wealth drift of agents in group $A$ and the wealth drift of the economy. The second term is an Ito correction term. The third term is due to the OLG setup. It is due to the difference between the average wealth of newborns $\phi_t$ and the average wealth of households in group $A$ that die, $x_t/\pi_A$. This OLG term ensures that $x_t$ is stationary: because the volatility of $x$, $\sigma_x$, is zero at the boundaries 0 and 1, and its drift $\mu_x$ is positive at 0 and negative at 1, no group of agents dominates the economy in the long run.

Market Price of Risk The second step of our basic mechanism is that, when more wealth falls into the hands of risk-tolerant households, stock prices increase and future returns are lower. To gain some intuition on this relationship in the model, I now consider the determination of the equilibrium price of risk $\kappa_t$.

Applying Ito’s lemma on Equation (15), one obtains that the wealth-weighted average of individual wealth volatility equals the total quantity of risk:

$$x_t\sigma_{W_{At}} + (1-x_t)\sigma_{W_{Bt}} = \sigma + \sigma_{p_t}$$ (20)

Substituting the volatility of individual wealth from Equation (11), one obtains the market price of risk in terms of individual RRA:

$$\kappa_t = \Gamma_t(\sigma + \sigma_{p_t}) - H_t$$ (21)

where $\Gamma_t$ corresponds to the aggregate RRA and $H_t$ corresponds to the aggregate hedging demand:

$$\frac{1}{\Gamma_t} = \frac{x_t}{\gamma_A} + \frac{1-x_t}{\gamma_B}$$ (22)

$$H_t = x_t H_{At} + (1-x_t) H_{Bt}$$ (23)
The market price of risk $\kappa_t$ is the product of the aggregate RRA $\Gamma_t$ times the total quantity of risk $\sigma + \sigma_{pt}$, minus the total demand for risk due to the hedging.

The aggregate RRA $\Gamma_t$ is a wealth-weighted harmonic mean of individual RRAs. The higher the share of wealth owned by the agents in group A, $x_t$, the lower the aggregate risk aversion $\Gamma_t$. Therefore, ignoring for a moment the hedging demand, an increase in the fraction hold by $x_t$ decreases the market price of risk $\kappa_t$.

**Risk-Free Rate** Applying Ito’s lemma on Equation (15), one obtains that the wealth-weighted average of individual wealth growth, plus a OLG term, equals the endowment growth:

$$x_t \mu_{A_t} + (1 - x_t) \mu_{B_t} + (\delta + \eta) (\phi_t - 1) = \mu + \mu_{pt} + \sigma \sigma_{pt} \tag{24}$$

The OLG term depends on the death rate and population growth times the difference between the relative wealth of newborn households $\phi$ and one. Intuitively, because the wealth of newborns does not equal the wealth of deceased households, there is a wedge between the average individual wealth growth and the aggregate growth.

Substituting the drift of individual wealth from Equation (12), one obtains the risk-free rate in terms of individual EIS:

$$r_t = \rho + \frac{1}{\Psi_t} \left[ \mu + \mu_{pt} + \sigma \sigma_{pt} - (\delta + \eta) (\phi_t - 1) - \left( x \frac{1 + \psi_A}{2 \gamma_A} + (1 - x) \frac{1 + \psi_B}{2 \gamma_B} \right) \kappa^2_t - \Phi_t \right] \tag{25}$$

where $\Psi_t$ corresponds to the aggregate EIS and $\Phi_t$ corresponds to aggregate changes in investment opportunities:

$$\Psi_t = x_t \psi_A + (1 - x_t) \psi_B \tag{26}$$
$$\Phi_t = x_t \Phi_A + (1 - x_t) \Phi_B \tag{27}$$

The higher the share of wealth owned by the agents in group A, $x_t$, the closer the aggregate elasticity of substitution to $\psi_A$. If agents in group A are also more willing to substitute inter-temporally, the risk-free rate tends to decrease as their wealth share $x_t$ increases.

**Volatility of Asset Prices** One reason we are interested in models with heterogeneous agents is that they can potentially explain the excess volatility of asset prices. As the share of wealth owned by agents in group A, $x_t$, fluctuates, both the market price of risk and the interest rate fluctuate, which generates fluctuations in the price-dividend ratio.
Consider $R_t$ the cumulative return of an asset that gives the flow of dividend $Y_t$. Denoting $p_{Dt}$ the price-dividend ratio of this asset, we can write:

$$\frac{dR_t}{R_t} = \left( r_t + \kappa_t \sigma_R t \right) dt + \left( \sigma + \sigma_{pD} \right) dZ_t$$

(28)

The excess volatility of the return is given by $\sigma_{pD}$, the geometric volatility of the price-dividend ratio. I propose a continuous-time version of the Campbell and Shiller (1988) decomposition for the volatility of the price-dividend ratio:

**Proposition 3** (Campbell-Shiller in Continuous-Time Model). The volatility of the price-dividend ratio, $\sigma_{pD}$, is given by:

$$\sigma_{pD} \approx - \int_0^{+\infty} e^{-\alpha t} E \left[ r_t | x_0 = x \right] \sigma_x \text{ Risk-Free Rate Channel} - \int_0^{+\infty} e^{-\alpha t} E \left[ \kappa_t \sigma_R t - \frac{1}{2} \sigma_{Rt}^2 | x_0 = x \right] \sigma_x \text{ Excess-Returns Channel} \quad (29)$$

with $\alpha = \exp(-E[\log(p^D)])$.

The volatility of the price-dividend ratio can be decomposed into two terms: the first term depends on the volatility of expectations of future risk-free rates. The second term depends on the volatility of expectations of future expected excess log-returns.

By exchanging the expectation and the derivative operations in the RHS of Equation (29), one obtains:

$$\sigma_{pD} \approx - \int_0^{+\infty} e^{-\alpha t} E \left[ \frac{\partial x_t}{\partial x_0} x_0 = x \right] \sigma_x \text{ Risk-Free Rate Channel} - \int_0^{+\infty} e^{-\alpha t} E \left[ \frac{\partial x_t}{\partial x_0} (\kappa \sigma_R t - \frac{1}{2} \sigma_{Rt}^2) x_0 = x \right] \sigma_x \text{ Excess-Returns Channel}$$

where $\partial x_t / \partial x_0$, denotes the first-variation of the process $x_t$. This equation expresses the risk-free rate channel directly in terms of the derivative of the risk-free rate around $x$, and the excess-returns channel in terms of the derivative of expected excess log-returns around $x$. This suggests that, the higher the derivative of the risk-free rate and of the market price of risk with respect to the group share $x$, the larger the volatility of asset prices.

\textsuperscript{15}Formally, the first-variation process $D_t = \partial x_t / \partial x_0$ associated to the diffusion $x_t$ satisfies:

$$\frac{dD_t}{D_t} = \frac{\partial \mu_x(x_t)}{\partial x} dt + \frac{\partial \sigma_x(x_t)}{\partial x} dZ_t \quad (31)$$

See for instance Fournié et al. (1999).
In turn, as shown in (21) and (25), the derivatives of the risk-free rate and of the market price of risk depend critically on the derivative of the aggregate RRA $\Gamma$ and of the inverse of the aggregate EIS $1/\Psi$ with respect to the group share $x$:

$$\frac{\partial \Gamma}{\partial x} = \Gamma^2 \left( \frac{1}{\gamma_B} - \frac{1}{\gamma_A} \right)$$  \hspace{1cm} (32)

$$\frac{\partial \Psi^{-1}}{\partial x} = \Psi^{-2}(\psi_B - \psi_A)$$  \hspace{1cm} (33)

Together, these equations relate the volatility of asset prices to the degree of preference heterogeneity across households. Importantly, these derivatives themselves decrease in $x$ (in absolute value); this suggests the volatility of asset prices in the model is higher for low value of the group share $x$.

An important question is whether the degree of households heterogeneity necessary to generate volatile asset prices in the model is consistent with the data. The next section examines how moments about the wealth distribution help discipline household heterogeneity.

## 4 Wealth Distribution

In this section, I examine the wealth distribution obtained in the model. Because households at the top of the distribution tend to be more exposed to aggregate risk, top wealth shares move with aggregate shocks and the distribution exhibits a fat tail, driven by the high average returns of top households.

### 4.1 Dynamics of Wealth Density

Denote $w_{it}$ the wealth of agent $i$ relative to the per-capita wealth in the economy, i.e. $w_{it} = W_{it}/(p_t Y_t)$. Applying Ito’s lemma, the law of motion of the relative wealth $w_{it}$ is

$$\frac{dw_{it}}{w_{it}} = \mu_{w_{it}} dt + \sigma_{w_{it}} dZ_t$$  \hspace{1cm} (34)

where $\mu_{w_{it}}$ and $\sigma_{w_{it}}$ are given by

$$\sigma_{w_{it}} = \sigma_{W_{it}} - \sigma - \sigma_{p_t}$$  \hspace{1cm} (35)

$$\mu_{w_{it}} = \mu_{W_{it}} - \mu - \mu_{p_t} - \sigma_{p_t} - \sigma_{w_{it}}(\sigma + \sigma_{p_t})$$  \hspace{1cm} (36)

I first characterize the dynamics of the wealth density in the model. Denote $g_{jt}$ the density of relative wealth within each group of agent $j \in \{A, B\}$, and $g_t$ the density of relative wealth across all households, i.e.

$$g_t = \pi_A g_{At} + (1 - \pi_A)g_{Bt}$$  \hspace{1cm} (37)
Finally, denote \( g_{\chi t} \) the distribution of relative wealth for newborn agents. The next proposition characterizes the law of motion of the wealth density within each group \( g_{jt} \) for \( j \in \{A,B\} \):

**Proposition 4** (Kolmogorov-Forward Equation with Aggregate Shocks). The law of motion of \( g_{jt} \) is given by

\[
dg_{jt} = \left( -\mu_{w_{jt}} \partial_w (wg_{jt}) + \frac{1}{2} \sigma_{w_{jt}}^2 \partial_{ww} (w^2 g_{jt}) + (\delta + \eta) (g_{\chi t} - g_{jt}) \right) dt - \sigma_{w_{jt}} \partial_w (wg_{jt}) dZ_t
\]

Given the evolution of individual wealth \((\mu_{w_{jt}} dt, \sigma_{w_{jt}} dZ_t)\), this equation gives the evolution of the wealth density \( g_{jt+dt} - g_{jt} \). The key difference from the existing literature is that, because households choose different exposures to aggregate shocks (i.e. \( \sigma_{w_{jt}} \neq 0 \)), the wealth density is stochastic.

### 4.2 Fat-Tail

While a full characterization of the entire wealth distribution is not feasible, I show that one can characterize analytically its right tail.

**Fat-Tail**  By analogy with the case of a static distribution, I define the tail index of a stochastic distribution as the smallest number \( \zeta \) for which moments of order higher than \( \zeta \) converge to infinity.

**Definition 1** (Fat-Tail). A density \( g_t \) is fat-tailed with tail index \( \zeta \) if there exists \( \zeta > 0 \) such that its moments of order \( \xi > 0 \) converge to infinity almost surely if and only if \( \xi \geq \zeta \), i.e.

\[
\lim_{t \to +\infty} \int_0^\infty w^\xi g_t(w)dw = +\infty \text{ a.s. iff } \xi \geq \zeta
\]

The next proposition characterizes the tail index of the wealth density in the economy.

**Proposition 5** (Tail Index). Assume that \( x_t \) is irreducible positive recurrent with unique invariant probability measure. Denote

\[
\zeta = \frac{\delta + \eta}{\mathbb{E}[\mu_{w_{At}} - \frac{1}{2} \sigma_{w_{At}}^2]}
\]

where \( \mathbb{E} \) denotes the expectation with respect to the invariant probability measure of \( x \). If the following conditions are satisfied:

\[\text{Denoting } g_{\chi} \text{ the density of } \chi_i \text{ defined in Section 3, we have}
\]

\[
g_{\chi t}(w) = \frac{1}{\phi_t} g_{\chi} \left( \frac{1}{\phi_t} w \right).
\]
1. Agents in group $A$ grow on average faster than the economy and that agents in group $B$, i.e.

$$E\left[\mu_{wAt} - \frac{1}{2}\sigma_{wAt}^2\right] > \max\left(0, E\left[\mu_{wBt} - \frac{1}{2}\sigma_{wBt}^2\right]\right) \quad (42)$$

2. The distribution of human capital $\chi_i$ has a tail thinner than $\zeta$.

Then the wealth distribution is fat-tailed with tail index $\zeta$.

The tail index of the wealth distribution $\zeta$ can be expressed as the ratio of the sum of the death rate and of the population growth rate, $\delta + \eta$, to the average logarithmic wealth growth rate of agents in group $A$ relative to the rest of the economy, $E[d\ln w_{At}] / dt = \mu_{wAt} - \frac{1}{2}\sigma_{wAt}^2$.

This formula generalizes the static model of Wold and Whittle (1957) along two dimensions. First, because the wealth growth of top households depends on the aggregate state of the economy, the tail index depends on the average wealth growth of top households. Second, because the relative wealth of top households is stochastic, the tail index depends on the average growth of their logarithmic wealth.

Proposition 5 shows that the distribution of labor income does not matter for the tail index of the wealth distribution, as long the right tail of the labor income distribution is thinner than the wealth distribution.$^{17,18}$ In other words, the tail index is robust to the exact distribution of labor income across households in the economy.

It is enlightening to compare this result to the case of infinite-horizon economies. In infinite-horizon economies, a stationary wealth distribution obtains only when the logarithmic relative wealth growth rate of agents in group $A$ is zero on average.$^{19}$ OLG models break this equivalence, by considering death and population growth. What Proposition 5 shows is that the growth rate of agents in group $A$ is still a key statistic: it controls how thick the right tail of the wealth distribution is.

**Law of motion of $x_t$**  As shown in Section 3, the law of motion of $x_t$ is directly related to the wealth growth of households of type $A$ relative to the economy. More precisely, applying Ito’s lemma on Proposition 2 gives:

$$E[d\ln x_t] = E[d\ln w_{At}] + (\delta + \eta) \left(\frac{\pi A}{x_t} \phi_t - 1\right) dt \quad (43)$$

$^{17}$A similar result obtains in deterministic economies. See, for instance, Gabaix et al. (2016).

$^{18}$This is the case empirically: the wealth distribution has a tail index of 1.5 while the distribution of labor income has a tail index between 2 and 3. See for instance Toda (2012).

$^{19}$See for instance Blume et al. (1992). To show it in this particular setup, set $\delta = \eta = 0$ in (43).
Because $x_t$ is ergodic, the drift of $\ln x_t$ must average to zero. This gives a relationship between the relative wealth growth of agent in group $A$ and the average level of $x$:

**Proposition 6** (Relation Between Tail Index and Group Share). *Under the assumptions of Proposition 5, the tail index $\zeta$ of the wealth distribution is given by:

$$
\zeta = \frac{1}{1 - E \left[ \frac{\pi_A \phi_t}{x_t} \right]}
$$

(44)

where $E$ denotes the expectation with respect to the stationary density of $x$. In particular $\zeta \geq 1$, i.e. the tail of the distribution is less thick than Zipf’s law.

The formula shows that the tail index $\zeta$ is directly related to $E \left[ \frac{\pi_A \phi_t}{x_t} \right]$, i.e. the average ratio of the relative wealth of a newborn agent ($\phi_t$) to the average wealth of an agent in group $A$ ($x_t/\pi_A$). Small changes in the tail index correspond to large changes in this ratio. For instance, a tail index close to 1 (i.e. Zipf’s law) corresponds to an economy where the average wealth of an agent in group $A$ is infinitely large compared to the average human capital of a newborn. In contrast, a tail index closer to 1.5 corresponds to an economy where the average wealth of an agent in group $A$ is only three times higher than the average human capital of a newborn. In short, small changes in the tail index correspond to large differences in the underlying economy.

**4.3 Dynamics of Top Wealth Shares**

I now integrate the Kolmogorov-Forward equation to obtain the law of motion of top wealth shares. This allows me to obtain the dynamics of top wealth shares in terms of the dynamics of individual wealth.

Let $\alpha$ be a top percentile, e.g. top 1%. Denote $q_t$ the $\alpha$—quantile, i.e.,

$$
\alpha = \int_{q_t}^{+\infty} g_t(w)dw
$$

(45)

$q_t$ corresponds to the wealth of an agent exactly at the $\alpha$ percentile threshold of the distribution.

We can now define the wealth share owned by the top percentile $\alpha$, $S_t$:

$$
S_t = \int_{q_t}^{+\infty} w g_t(w)dw
$$

(46)

The following proposition characterizes the dynamics of the wealth share of the top percentile $\alpha$:

**Proposition 7** (Law of Motion of Top Wealth Shares). *The law of motion of the top wealth share $S_t$ is

$$
\frac{dS_t}{S_t} = \mu_S dt + \sigma_S dZ_t
$$

(47)
where $\mu_{St}$ and $\sigma_{St}$ are given by

\[
\sigma_{St} = E^{gw}[\sigma_{w_{it}} | w_{it} \geq q_t] \\
\mu_{St} = E^{gw}[\mu_{w_{it}} | w_{it} \geq q_t] + \frac{q_t g_t(q_t)}{2S_t} \text{Var}^{gt}[\sigma_{w_{it}} | w_{it} = q_t] \\
+ (\delta + \eta) \left( \frac{\alpha q_t}{S_t} - 1 \right) + \frac{\delta + \eta}{S_t} \int_{q_t}^{\infty} (w - q_t) g_{wh}(w) dw
\]

where the expectation $E^{gw}$ refer to the wealth-weighted, cross-sectional average with respect to the wealth density $g_t$.

The geometric volatility of the top wealth share, $\sigma_{St}$, is the wealth-weighted average geometric volatility of individuals in the top percentile. As $\alpha$ denotes an increasingly high percentile, there are relatively more agents of type A in the top percentile, and therefore the exposure of the wealth share of top percentiles converges to $\sigma_{w_{At}}$. The model therefore generates the empirical evidence found in Section 2 (Table 1).

The geometric drift of the top wealth share, $\mu_{St}$, is the sum of three terms. The first term corresponds to the average, wealth-weighted, geometric drift of individuals at the top. The second term is due to the heterogeneous exposure of households at the threshold. It depends on the variance of risk exposures across households at the quantile $q_t$. When a negative shock hits the economy, top wealth shares decrease a bit less than the wealth of households inside the top percentile, because some households from group B enter the top. Conversely, when a positive shock hits the economy, top wealth shares increase a bit more than the wealth of households inside the top percentile, because some households from group A enter the top. Therefore, heterogeneous exposure to aggregate shocks tends to increase the growth of top wealth shares due to a composition effect. The third term is due to the death of individuals at the top, as well as the eventual birth of households with human capital higher than $q_t$.

In particular, while changes in the composition of households in the top affect the average growth of top wealth shares, they do not change the exposure of top wealth shares to stock market returns. In other words, the exposure of top wealth shares to aggregate shocks, as measured in Section 2, recovers the average exposure of individual households inside the top percentile.

\[E^{gw} \text{ denotes the wealth-weighted average and } \text{Var}^{gt} \text{ denotes the variance with respect to the wealth density } g_t.\]
5 Estimating the Model

I now bring the model to the data. Qualitatively, the model generates key stylized facts about wealth inequality and the wealth distribution (i) top wealth shares increase when stock market returns are high (ii) top wealth shares predict future excess-returns (iii) the wealth distribution is fat-tailed. However, I highlight a key tension between matching quantitatively asset prices and the wealth distribution: to match the high volatility of asset prices, the model requires a wealth distribution with a tail much thicker than the data.

5.1 Estimation Method

Method  I estimate the parameters of the model by minimizing the distance between moments from the data and those implied by the model. I proceed as follows. I select a vector of moments $m$ computed from the actual data. Given a candidate set of parameters $\Theta$, I solve the model, and compute the moments $\hat{m}(\Theta)$. I search the set of parameters $\hat{\Theta}$ that minimizes the weighted deviation between the actual and simulated moments $J(\Theta)$, i.e.

$$J(\Theta) = (m - \hat{m}(\Theta))' W (m - \hat{m}(\Theta))$$

$$\hat{\Theta} = \arg\min_{\Theta} J(\Theta)$$

where $W$ is a weight matrix, computed as inverse of the variances of moments as measured in the data.

Asset Prices Moments  Following Gärleanu and Panageas (2015), I use four asset price moments, corresponding to the average and standard deviation of the risk-free rate and of stock market returns (in real term). The data for the average equity premium, the volatility of returns, and the average interest rate are from Shiller (2015). The volatility of the real risk-free rate is inferred from the yields of 5-year constant maturity TIPS following Gärleanu and Panageas (2015).

Wealth Inequality Moments  Compared to Gärleanu and Panageas (2015), I add two moments about the wealth distribution.

The first moment is the elasticity of top wealth shares to stock market returns. As shown in Proposition 7, the moment measures the risk exposure of households in group $A$ relative to the economy. Empirically, following the evidence in Table 1, this moment equals 0.48.

The second moment is the tail index of the wealth distribution. As shown in Proposition 5, the moment measures the average logarithmic wealth growth rate of households in group $A$ relative to
the economy. Using a maximum likelihood estimate, I measure a tail index of $\zeta = 1.5$, consistent with previous studies.\footnote{See, for instance, Klass et al. (2006) or Vermeulen (2018).} Graphically, Figure 3 plots the log percentile as a function of the log net worth for the U.S. distribution, in the SCF and in Forbes 400 data. The linear slope is characteristic of a distribution with a Pareto tail.

As seen in Section 4, these two moments about the wealth distribution discipline respectively the geometric volatility and the geometric growth of individuals in group $A$.\footnote{To make the exercise more transparent, I do not add the estimate of the predictability regressions as a targeted moment. As shown in Section 2, the moment has large standard errors, and its effect on the estimation procedure is minimal.}

**Calibrated Parameters** The law of motion of the endowment process per capita is $\mu = 2\%$ and $\sigma = 4.1\%$. The death rate is $\delta = 2\%$. The population growth rate is $\eta = 1\%$.

Following the approach of Barro (2006), I report the stock market returns for a firm with a debt-equity ratio equal to the historically observed debt-equity ratio for the U.S. non financial corporate sector, using a debt-equity ratio equal to $\lambda \approx 0.5$. Finally, human capital is calibrated based on U.S. data about life-cycle labor income, as in Gârleanu and Panageas (2015).\footnote{See Appendix D.}

**Estimated Parameters** The model has 6 remaining parameters that I estimate. 2 parameters correspond to the preference parameters specific to households in group $A$ ($\gamma_A, \psi_A$) and 2 parameters correspond to the preference parameters specific to households in group $B$ ($\gamma_B, \psi_B$). I impose sensible restrictions on the preference parameters of agents in group $B$, i.e. $\psi_B \geq 0.05$ and $1/\gamma_B \geq 0.05$. The remaining parameters is the subjective discount rate $\rho$ and the population share of the agents in group $A$, $\pi_A$. The model has 6 moments to estimate these 6 parameters: 4 moments about asset prices, and 2 moments about the wealth distribution.

### 5.2 Results

I now bring the model to the data. Qualitatively generates the stylized facts about the joint dynamics of wealth inequality and asset prices documented in Section 2. However, to generate volatile asset prices, the model requires a wealth distribution with a tail much thicker than the data: there is a key tension between matching asset prices and the wealth distribution.

**Estimation on Asset Prices Only** I first estimate the model on asset price moments, which is a very similar compared to to Gârleanu and Panageas (2015). Column (2) of Table 5 reports the
result of the estimation. Figure 4 plots the market price of risk, the interest rate, the stationary
density of the state variable, and the Campbell-Shiller decomposition of the volatility of the price-
dividend ratio.\textsuperscript{24} The model matches very well asset price moments: it generates a high equity
premium, with a high volatility, together with a low risk-free rate with low volatility.

The model can match the high volatility of asset prices because it has a high degree of preference
heterogeneity, both in terms of RRA ($\gamma_A \approx 1.5$ vs $\gamma_B \approx 17$) and in terms of EIS ($\psi_A \approx 0.8$ vs $\psi_B = 0.05$). Due to this high degree of preference heterogeneity, the market price of risk and the
risk-free rate are very sensitive to the wealth share of group $A$ (as explained in Section 3). Indeed,
Figure 4 show that both prices decrease sharply in the wealth share of group $A$. This sensitivity,
combined with the fact that the model spends a large amount of time in region where $x$ is low, i.e.
where the sensitivity of these prices is high, generates a large volatility of asset prices: Figure 4
plots the excess volatility of returns, as a sum of a risk-free rate channel and an excess-returns
channel. Overall, 75% of excess volatility is due to the expected excess-return channel, while 25%
is due to the risk-free rate channel. The possibility that agents in group $A$ may be “wiped out”
after a series of negative shocks, i.e. that their wealth share $x$ may approach zero, drives a large
amount of asset prices volatility, even in normal times.

I now assess whether the model fits moments about the wealth distribution, that were not
targeted in the first estimation. I find that the model overestimates the exposure of top wealth
shares to stock market returns: it is 0.64 in the model compared to 0.48 in the data. More
importantly, the model overestimates the thickness of the tail of the distribution: in the model, the
tail index of the wealth distribution is 1.1, whereas it is close to 1.5 in the data. Visually, Figure 3
shows that the right tail of the wealth distribution is much thicker in the model than the data.

In the model, agents in group $A$ grow much faster than the rest of the distribution, which
leads to a wealth distribution with a thick tail (Proposition 5). As seen in Proposition 6, this
thick tail corresponds to a high ratio of the wealth share of agents in group $A$ relative to their
population share. Indeed, in the model, while the population share of agents in group $A$ is only
1%, their wealth share hovers around 20% (i.e. agents in group $A$ are 20 times more wealthy than
the average household).

\textbf{Adding Exposure of Top Shares Only} I re-estimate the model targeting only asset prices
and the exposure of top wealth shares to stock market returns. I show that the model fits relatively

\textsuperscript{24}The only difference is that I consider an economy where population grows at rate 1%. That being said, the result
of the estimation is very similar to Gärleanu and Panageas (2015), both in terms of preference parameters and in
terms of moments.
well all targeted moments, but still vastly overestimates the right tail of the distribution.

Column (3) of Table 5 estimates the model on asset prices and the exposure of top wealth shares. The estimated model matches relatively well asset prices and the exposure of top wealth share. In particular, the model matches the high volatility of asset prices: the exposure of top wealth shares to stock market returns is 0.51 in the model, compared to 0.48 in the data.

As discussed above, targeting the exposure of top wealth shares disciplines the RRA of households in group A, \( \gamma_A \). To maintain a high volatility of asset prices despite the lower degree of RRA heterogeneity, the model now exhibits a larger degree of EIS heterogeneity. Correspondingly, the table reports that a larger share of the total volatility of asset prices is now driven by changes in the risk-free rate, from 25% in the previous estimation, to 35% in the current estimation.

While the model matches the targeted moments well, the resulting wealth distribution still has a right tail much thicker than the data, with a tail index of 1.1, compared to 1.5 in the data. As in the previous estimation of the model, there is still a large difference between the average wealth growth of agents in group A compared to group B, which generates a wealth distribution with a tail thicker than the data.

**Adding Tail Index** To ask whether the model can jointly match asset prices and the wealth distribution, I re-estimate the model in Column (4) of Table 5, targeting jointly the four asset prices moments and the two moments about the wealth distribution, i.e. the exposure of top wealth shares to stock market returns. The model cannot jointly match asset prices and the wealth distribution: the J-statistic, which measures the distance between the model and the data, jumps from 0.4 to 15.

The reason the model fails to match both sets of moments is that there is a tension between matching the volatility of asset prices and the tail index of the wealth distribution. To match a thinner tail index, the model needs to decrease the average wealth growth of households in group A (either by increasing their subjective discount rate \( \rho \) or by decreasing their EIS \( \psi_A \)). Because the wealth-weighted average growth of individual households has to sum up to aggregate wealth growth, this must be compensated either by an increase of the average wealth growth of agents in group B, or by an increase in the average group share of agents in group A. In either case, the average sensitivity of aggregate RRA \( \Gamma \) and of the inverse of the aggregate EIS \( 1/\Psi \) to the group share \( x \) must decrease. In turn, this decreases the average sensitivity of the market price.

\(^{25}\)See Equation (24).

\(^{26}\)Indeed, Equation (32) shows that the sensitivity of aggregate RRA and of the inverse of the aggregate EIS is high when \( x \) is close to zero and when \( \gamma_B \) and \( 1/\psi_B \) are high.
of risk and of the risk-free rate to the group share, which ultimately decreases the volatility of asset prices.\textsuperscript{27}

Column (4) reports the results of model estimated on all moments. The model overestimates the interest rate (3.2\% in the model compared to 2.8\% in the data). This is because, in the model, both agents have a very low EIS. Moreover, the model underestimates the equity premium (3.6\% in the model compared to 5.2\% in the data). The reason the model misses the equity premium, rather than the standard deviation of returns, is that this moment is less precisely estimated, and therefore is given less weight in the J-statistic. Moreover, having such a low equity premium allows the model to lower the average wealth growth of households in group $A$, by decreasing their compensation for holding aggregate risk. In other words, underestimating the equity premium allows the model to match the thinner tail of the wealth distribution, without substantially changing the degree of preference heterogeneity. Finally, even though this degree of heterogeneity still generates a substantial volatility in asset prices, half of the fluctuations in asset prices are now driven by fluctuations in expected risk-free rate, rather than fluctuations in expected excess stock returns, which is counter-factual.\textsuperscript{28}

5.3 Extensions

The environment is highly stylized and one could consider additional ingredients that would make the wealth distribution more realistic. In this section, I explore three extensions to the standard model that do not change my main conclusion.

**Idiosyncratic Volatility** In the baseline model, there are no idiosyncratic wealth shocks: the position of households in the wealth distribution only depends on their group, their age, and their initial human capital. Yet, in the data, idiosyncratic wealth shocks play an important role in driving the right tail of the distribution.\textsuperscript{29}

Would incorporating this idiosyncratic volatility help the model to match the data? In the appendix, I show that adding idiosyncratic wealth volatility to the model systematically thickens the right tail of the distribution (Lemma 3). Since the baseline model already gives a right tail that is too thick compared to the data, adding idiosyncratic wealth shocks only worsens the tension between matching asset prices and the right tail of the distribution. Formally, Column (2) Table 6

\textsuperscript{27}As seen in Proposition 3.
\textsuperscript{28}See, for instance, Campbell and Shiller (1988).
\textsuperscript{29}See, for instance, Benhabib et al. (2015a). For recent asset prices models where idiosyncratic wealth shocks play an important role, see Kogan et al. (Forthcoming) and Gärleanu and Panageas (2017).
re-estimates the model with an idiosyncratic wealth volatility of 10%. For simplicity, I assume that the presence of these idiosyncratic shocks does not affect the saving decisions of households.\textsuperscript{30} As predicted, the fit worsens: the J-statistic increases to 21.

**Heterogeneity in Subjective Discount Rates** While the model considers heterogeneity in RRA and EIS, households have the same subjective discount rate ($\rho$). One may wonder whether considering heterogeneity in subjective discount rate $\rho$ could help the model better fit the data. To check this, I report the result of the estimated model in Column (3) of Table 6. I find very similar estimates compared to the model where agents have homogeneous discount rates. This is because, in the estimated model, the EIS of agents in group B $\psi_B$ is so low that the exact value of their subjective discount rate $\rho_B$ has very little impact on their saving decision (Proposition 1), and therefore on aggregate quantities.

**Bounded Lives** To get analytical results on the tail index of the wealth distribution, the model assumes that the death probability of households does not depend on age. In particular, this means that, in the model, certain households live for a large amount of time and end up with a disproportional amount of wealth compared to the rest of the economy. In reality, households usually die after a certain age, which limits how rich richer households can really be.

In a static model, where the wealth distribution is in a steady state, one can show that this assumption has no impact on the tail index of the wealth distribution.\textsuperscript{31} This is because the tail index is a local property, that characterizes the slope of the wealth density. Imposing households to die after a certain age only results in a truncated Pareto distribution, with the same tail index as the original model.

To check that this still holds true in a model with aggregate shocks, I rely on simulations. More precisely, I simulate the wealth distribution in the model, with the added assumption that households older than 100 years die with probability one. I find that incorporating this more realistic “upper bound” on age does not change the tail index of the wealth distribution, measured as the average slope of the wealth density with respect to wealth.\textsuperscript{32}

\textsuperscript{30}Precautionary saving would tend to increase the saving rate of households, which corresponds to a rise in their subjective discount rate $\rho$.

\textsuperscript{31}See, for instance, Steindl (1965)

\textsuperscript{32}A video of the wealth distribution in an economy with fluctuating drift is available at http://www.matthieugomez.com/wealthinequality.html. Even though the wealth distribution is not in a steady state, the slope of the log density with respect to log wealth remains stable over time.
6 Conclusion

The results of this paper depict a strong interplay between asset prices and wealth inequality. Because rich households hold more risky assets, realized stock returns generate fluctuations in wealth inequality over time. Conversely, in periods of high inequality, more wealth is in the hands of rich households, the risk-tolerant investors. Therefore, risk premia are low: a high level of inequality predicts low future returns. This interplay is at the heart of heterogeneous agents asset pricing models. I have shown that these models can qualitatively account for these facts. However, the standard models tend to overestimate the thickness of the tail of the wealth distribution.

There has been a growing interest in models with heterogeneous agents in recent year. These models tend to assume on a substantial amount of heterogeneity between households, which is hard to measure empirically. This paper shows that certain moments about the wealth distribution help discipline the degree of heterogeneity present in the data. In particular, this paper focuses on the tail exponent of the wealth distribution, since it captures the wealth dynamics of top households and it is robust to the exact specification of the model.

The implications of my analysis extend beyond asset pricing. The interplay I put forward can have effects on real quantities as well, through two channels. First, because the level of inequality affects the cost of capital, this can lead to changes in corporate investment policies. Second, a recent literature has also emphasized the role of inequality for aggregate demand (Mian et al. (2013), Kaplan et al. (2016)). Exploring these channels requires moving away from an endowment economy, which I leave for future research.
References


Elliott, Graham and James H Stock, “Inference in Time Series Regression when the Order of Integration of a Regressor is Unknown,” Econometric Theory, 1994, 10 (3-4), 672–700.


Table 1: The Exposure to Stock Returns Increases Across the Wealth Distribution

<table>
<thead>
<tr>
<th></th>
<th>Flow of Funds</th>
<th>Kopczuk and Saez (2004)</th>
<th>Forbes 400</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
</tr>
<tr>
<td></td>
<td>(4)</td>
<td>(5)</td>
<td></td>
</tr>
<tr>
<td><strong>Panel A: Wealth</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Excess Stock Returns</td>
<td>0.48***</td>
<td>0.64***</td>
<td>0.81***</td>
</tr>
<tr>
<td></td>
<td>(0.11)</td>
<td>(0.17)</td>
<td>(0.15)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.27</td>
<td>0.22</td>
<td>0.36</td>
</tr>
<tr>
<td>$N$</td>
<td>53</td>
<td>53</td>
<td>53</td>
</tr>
</tbody>
</table>

**Panel B: Wealth Share**

|                      |              |                         |            |
|                      | (2)          | (3)                     | (4)        |
|                      | (5)          |                         |            |
| Excess Stock Returns | 0.17**       | 0.34***                 | 0.48***    | 0.46**     |
|                      | (0.07)       | (0.07)                  | (0.11)     | (0.19)     |
| $R^2$                | 0.20         | 0.39                    | 0.23       | 0.13       |
| $N$                  | 53           | 53                      | 53         | 31         |

Notes: This table reports the results of the regression of the excess wealth growth of households in a given percentile group on the excess stock returns, i.e. Equation (1):

$$\log \left( \frac{W_{G,t+3}}{W_{G,t-1}} \right) - 4 \log R_{ft} = \alpha_G + \beta_G (\log R_{Mt} - \log R_{ft}) + \epsilon_{Gt}$$

The dependent variable is the growth of wealth in Panel A and the growth of wealth shares in Panel B. Each column corresponds to a different group of households. The first column corresponds to all U.S households. Columns (2) to (4) correspond to increasing top percentiles in the wealth distribution, using data from Kopczuk and Saez (2004). Columns (5) correspond to the Top 0.0003%; the percentiles are chosen so that the group include the 400 wealthiest individuals in 2015. Estimation via OLS. Standard errors in parentheses and estimated using Newey-West with 3 lags. *, **, *** indicate significance at the 0.1, 0.05, 0.01 levels, respectively.
Table 2: The Exposure to Stock Returns Across the Wealth Distribution: Controlling for Composition

<table>
<thead>
<tr>
<th></th>
<th>Forbes 400</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>All</td>
</tr>
<tr>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>Excess Stock Returns</td>
<td>0.93***</td>
</tr>
<tr>
<td></td>
<td>(0.25)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.31</td>
</tr>
</tbody>
</table>

Notes: In the first column, the table regresses the total growth of the wealth in the top 400 on excess stock market returns, as in Table 1. In the second column, the table regresses the “within growth” on excess stock market returns. The “within growth” corresponds to the yearly growth of households in the top 400, whether or not they drop out of the top by the end of the year. See Gomez (2018) for more details on the construction of the “within” term.

Table 3: The Share of Wealth Owned by the Top 0.01% And Future Excess-Returns

$$\sum_{1 \leq h \leq H} \log R^d_{t+h} - \log R^f_{t+h} = \alpha + \beta_H \text{Log Top Wealth Shares}_t + \gamma_H X_t + \epsilon_{tH}$$

<table>
<thead>
<tr>
<th></th>
<th>Excess-Returns at Horizon $H = 1$</th>
<th>Excess-Returns at Horizon $H = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\beta_1$</td>
<td>$\gamma_1$</td>
</tr>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
</tr>
<tr>
<td>Log Top Share</td>
<td>-0.109*</td>
<td>0.031</td>
</tr>
<tr>
<td>Δ Log Top Share (5 years difference)</td>
<td>-0.271**</td>
<td>0.066</td>
</tr>
<tr>
<td>Log Top Share and Linear Trend</td>
<td>-0.264***</td>
<td>-0.003*</td>
</tr>
<tr>
<td>Log Top Share and Dividend Price</td>
<td>-0.172**</td>
<td>0.116*</td>
</tr>
<tr>
<td>Log Top Share and Dividend Payout</td>
<td>-0.141**</td>
<td>0.094</td>
</tr>
</tbody>
</table>

Notes: The table reports the result of the regressions of future excess-returns on the share of wealth owned by the Top 0.1% (row 1). Each row corresponds to a different set of regressors. Columns (1) (2) (3) report the results when the dependent variable is the one year excess-return. Columns (4) (5) (6) report the results when the dependent variable is the three-year excess-return. Estimation via OLS. Standard errors in parentheses and estimated using Newey-West with 3 lags. *, **, *** indicate significance at the 0.1, 0.05, 0.01 levels.
Table 4: The Share of Wealth Owned by the Top 0.01% And Future Excess-Returns
(Campbell and Yogo (2006) test)

<table>
<thead>
<tr>
<th>Case with $\rho = 0.89$</th>
<th>Case with $\rho = 1.02$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>$\bar{\beta}$</td>
</tr>
<tr>
<td>Log Top Share</td>
<td>$-0.39$</td>
</tr>
</tbody>
</table>

Notes: The time period is 1917-1951, the longest period where the wealth share of the top 0.01% is available without missing years. This table uses the test developed Campbell and Yogo (2006), that jointly takes into account the persistence of the predictor as well as its correlation with stock returns to compute the 90% confidence interval for $\beta$. The autoregressive lag length for the DF-GLS statistic is estimated to be 1, using the Bayes information criterion (BIC).

Table 5: Estimation

<table>
<thead>
<tr>
<th>Data</th>
<th>Model</th>
<th>$\beta$</th>
<th>$\beta + \zeta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
</tr>
<tr>
<td><strong>Parameters</strong></td>
<td>RRA $\gamma_A$</td>
<td>1.5</td>
<td>1.7</td>
</tr>
<tr>
<td></td>
<td>RRA $\gamma_B$</td>
<td>17</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>EIS $\psi_A$</td>
<td>0.8</td>
<td>0.85</td>
</tr>
<tr>
<td></td>
<td>EIS $\psi_B$</td>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td></td>
<td>Discount Rate $\rho$</td>
<td>0.1%</td>
<td>0.1%</td>
</tr>
<tr>
<td></td>
<td>Population share $\pi_A$</td>
<td>1%</td>
<td>1.5%</td>
</tr>
<tr>
<td><strong>Moments</strong></td>
<td>Equity Premium</td>
<td>5.2%</td>
<td>5.1%</td>
</tr>
<tr>
<td></td>
<td>STD Market Return</td>
<td>18.2%</td>
<td>18.1%</td>
</tr>
<tr>
<td></td>
<td>Average interest rate $R_f$</td>
<td>2.8%</td>
<td>2.7%</td>
</tr>
<tr>
<td></td>
<td>STD interest rate $R_f$</td>
<td>0.9%</td>
<td>0.7%</td>
</tr>
<tr>
<td></td>
<td>$\beta_{R_{M1}, R_{F1}} \rightarrow \Delta \log S_t$</td>
<td>0.48</td>
<td>0.64</td>
</tr>
<tr>
<td></td>
<td>Tail Index $\zeta$</td>
<td>1.5</td>
<td>1.1</td>
</tr>
<tr>
<td></td>
<td>$\beta_{\log S_t \rightarrow R_{M1+1}, R_{F1+1}}$</td>
<td>$-0.1$</td>
<td>$-0.08$</td>
</tr>
<tr>
<td></td>
<td>Risk-Free Rate Channel %</td>
<td>25%</td>
<td>35%</td>
</tr>
<tr>
<td></td>
<td>J-statistic</td>
<td>0.05</td>
<td>0.41</td>
</tr>
</tbody>
</table>

Notes: Column (1) corresponds to the moments in the data. Columns (2) to (4) correspond to different estimations of the baseline model presented in Section 5. All estimations include asset price moments but differ with respect to the choice of other moments. Column (2) reports the model estimated on asset prices only. Column (3) reports the model estimated on asset prices and the exposure of top wealth shares to stock market returns. Column (4) reports the model estimated on asset prices, the exposure of top wealth shares to stock market return, and the tail index $\zeta$. 
<table>
<thead>
<tr>
<th>Parameters</th>
<th>Data</th>
<th>Model</th>
<th>Model</th>
<th>( \rho_A \neq \rho_B )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td></td>
</tr>
<tr>
<td>Parameters</td>
<td>RRA ( \gamma_A )</td>
<td>1.4</td>
<td>1.5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>RRA ( \gamma_B )</td>
<td>17</td>
<td>17</td>
<td></td>
</tr>
<tr>
<td></td>
<td>EIS ( \psi_A )</td>
<td>0.6</td>
<td>0.7</td>
<td></td>
</tr>
<tr>
<td></td>
<td>EIS ( \psi_B )</td>
<td>0.05</td>
<td>0.05</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Discount Rate ( \rho_A )</td>
<td>0.1%</td>
<td>0.1%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Discount Rate ( \rho_B )</td>
<td>2%</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Population share ( \pi_A )</td>
<td>11.2%</td>
<td>9.0%</td>
<td></td>
</tr>
<tr>
<td>Moments</td>
<td>Equity Premium</td>
<td>5.2%</td>
<td>3.4%</td>
<td>3.6%</td>
</tr>
<tr>
<td></td>
<td>STD Market Return</td>
<td>18.2%</td>
<td>16.3%</td>
<td>16.2%</td>
</tr>
<tr>
<td></td>
<td>Average interest rate ( R_f )</td>
<td>2.8%</td>
<td>3.7%</td>
<td>3.1%</td>
</tr>
<tr>
<td></td>
<td>STD interest rate ( R_f )</td>
<td>0.9%</td>
<td>1.0%</td>
<td>0.8%</td>
</tr>
<tr>
<td>( \beta_{R_{M_t} - R_f \rightarrow \Delta \log S_t} )</td>
<td>0.48</td>
<td>0.46</td>
<td>0.50</td>
<td></td>
</tr>
<tr>
<td>Tail Index ( \zeta )</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td></td>
</tr>
<tr>
<td>( \beta_{\log S_t \rightarrow R_{M_{t+1}} - R_{f_{t+1}}} )</td>
<td>-0.1</td>
<td>-0.03</td>
<td>-0.06</td>
<td></td>
</tr>
<tr>
<td>Risk-Free Rate Channel %</td>
<td>56%</td>
<td>49%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>J-statistic</td>
<td>21.0</td>
<td>15.0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Notes: Column (1) corresponds to the moments in the data. Columns (2) to (3) correspond to different estimations of the extended model. All models are estimated on six moments: four moments about asset prices, and two moments about the wealth distribution (the exposure of top wealth shares to stock market returns and the tail index). Column (2) reports the model estimated with idiosyncratic volatility \( \nu \approx 10\% \). Column (3) reports the model estimated with heterogeneous subjective discount rates.
Figure 1: Exposure $\beta$ of Top Wealth Group to Stock Market Returns at Different Horizons

(a) Estate Tax Returns

(b) Forbes 400

Notes: This figure reports the results of regressing excess wealth growth of households in the top 0.01% (left) and in the top 400 (right) on excess stock returns i.e.

$$\log \left( \frac{W_{G,t-1+h}}{W_{G,t-1}} \right) - h \log R_{ft} = \alpha_{Gh} + \beta_{Gh} (\log R_{Mt} - \log R_{ft}) + \epsilon_{Ght}$$

Each figure reports the estimates for $\beta_{Gh}$, from $h = 0$ to $h = 8$, as well as the 5%-95% confidence interval using Newey-West with 3 lags.

Figure 2: The Wealth Share of the Top 0.01% and Average Excess-Returns

Notes: The figure plots the wealth share of the top 0.01% (log) and the 8-year sum of future excess-returns (opposite of). All series are normalized to have a standard deviation of one.
Figure 3: The Right Tail of the Wealth Distribution

Notes: The figure compares the log net worth (relative to the average net worth) to the log percentile in SCF and Forbes. More precisely, the figure plots the average log net worth within 40 logarithmically spaced percentile bins in SCF. The figure plots the average log net worth for each position in Forbes 400. The (opposite of) the slope estimate gives $\zeta \approx 1.5$ for SCF and for Forbes 400. In red, the figure plots the distribution implied by the model estimated on asset prices (see Table 5).
Figure 4: Asset Prices in the Model

(a) EIS and RRA

(b) Interest Rate and Market Price of Risk

(c) Stationary Density

(d) Campbell-Shiller Decomposition

Notes: The figure plots the interest rate, the market price of risk, the stationary density, and the Campbell-Shiller Decomposition of Proposition 3 for the model estimated on asset prices only (Columns (2) in Table 5). The Campbell-Shiller Decomposition is estimated by constructing $E[r_t | x_0 = x]$ and $E[\kappa_t \sigma_R^2 - \frac{1}{2} \sigma_R^2 R_t | x_0 = x]$ using Kolmogorov-Backward equation.
Appendix A

Cross Sectional Evidence To understand better what drives the heterogeneous exposure of top households to aggregate shocks, I also examine the heterogeneity in equity holding across the wealth distribution using the Survey of Consumer Finances (SCF). The survey is a repeated cross-section of about 4,000 households per survey year, including a high-wealth sample. The survey is conducted every three years, from 1989 to 2013. The respondents provide information on their networth, including their investment in public and private equity. I define the equity share as the total investment in equity over networth. I define the set of entrepreneurs as the households with equity held in an actively managed business.\footnote{The definition follows Moskowitz and Vissing-Jørgensen (2002).}

Figure A1a plots the average equity share within percentile bins across the wealth distribution. The average equity share of 0.4 masks a substantial heterogeneity across households. The equity share is essentially flat at 0.2 over the majority of the wealth distribution, but increases sharply within the top 1%. Figure A1b plots the equity share with respect to the log top percentiles, showing that the equity share is approximately linear in the log percentile at the top of the distribution. The figure suggests that the bulk of the heterogeneity is concentrated within the top percentiles, which justifies my focus on the top percentile.

A stylized fact in the household finance literature is that stock market participation increases with wealth (Vissing-Jørgensen (2002)). Therefore, the increase in the equity share within the top percentiles could be driven by an increase in the proportion of stockholders (i.e. the extensive margin). However, Panel B of Table A1 shows that the percentage of stockholders is constant within the top percentiles (90%). The increase in the equity share is entirely driven by the increase within stockholders. While the heterogeneity between stockholders and non-stockholders generates a lot of variations at the bottom of the wealth distribution, these variations account for a small share of total wealth.

Investment in risky assets comes mainly in two forms: public equity and private equity. Panel A of Table A1 decomposes the increase in equity share across the top percentiles between the two types of equity. The decomposition reveals that the increase in the equity share is mostly driven by an increase in the share of wealth invested in private equity. Panel C of Table A1 shows that the proportion of entrepreneurs increases sharply in the top percentile: the proportion of households with an actively managed business is 78.5% in the Top 0.01%, compared to 10% in the general population.\footnote{Similarly, Hurst and Lusardi (2004), using the Panel Study of Income Dynamics (PSID), show that the propensity of entrepreneurship increases sharply with wealth in the top percentiles.} The wealth of these entrepreneurs is mostly invested in their private business. A potential concern is that, if entrepreneurs cannot trade or sell their firms easily, the heterogeneity in private equity holdings may have no impact on stock market prices. However, Panel C of Table A1 shows that entrepreneurs hold substantial amounts of public equity (15%). Even with illiquid businesses, entrepreneurs can adjust their overall risky holdings at the margin.

Saez-Zucman series Saez and Zucman (2016) have recently proposed a new series for top wealth shares, which relies on Income Tax Returns. In Table A2, I estimate the stock market exposure of top...
wealth percentiles using this series. I find that the estimates are now uniformly lower compared to Kopczuk and Saez (2004). For instance, the stock market exposure of the Top 0.01% is 0.66 using Income Tax Returns, compared to 0.95 using estate tax returns or Forbes. This suggests that the methodology used in Saez and Zucman (2016) cannot capture business cycle frequencies of wealth shares, even though they may capture more accurately the long run fluctuations in inequality. Indeed, a certain number of wealth categories are constructed using trends and interpolations across years, which tends to bias down the estimates.

Appendix B

B.1 Proof

Proof of Proposition 1. The HJB equation associated with household’s problem is

\[ 0 = \max_{C_{jt}, \sigma_{Wjt}} \left\{ f(C_{jt}, V_{jt}) + E[dV_{jt}] \right\} \]  

(A1)

Given the homotheticity assumptions, the value function of the households in group \( j \in \{A, B\} \) with wealth \( W \) can be written:

\[ V_{jt}(W) = \frac{W^{1-\gamma_j} \sigma_{Wjt}^{-\gamma_j}}{1-\gamma_j p_{jt}} \]  

(A2)

Applying Ito’s lemma on HJB equation:

\[ 0 = \max_{C_{jt}, \sigma_{Wjt}} \left\{ \frac{1}{1-\psi_j} \left( \frac{C_{jt}^{1-\frac{1}{\psi_j}}}{W_{jt}^{1-\frac{1}{\psi_j} \sigma_{Wjt}^{\psi_j}}} - (\rho + \delta) \right) + \frac{1}{2(\psi_j - 1)} \sigma_{pjt}^2 + \frac{1}{\psi_j - 1} \sigma_{Wjt} \sigma_{pjt} \right\} \]  

(A3)

Substituting the expression for the wealth drift \( \mu_j \) using the budget constraint and dividing by \( 1 - \gamma_j \)

\[ 0 = \max_{C_{jt}, \sigma_{Wjt}} \left\{ \frac{1}{1-\psi_j} \left( \frac{C_{jt}^{1-\frac{1}{\psi_j}}}{W_{jt}^{1-\frac{1}{\psi_j} \sigma_{Wjt}^{\psi_j}}} - (\rho + \delta) \right) + \frac{1}{2(\psi_j - 1)} \sigma_{pjt}^2 + \frac{1}{\psi_j - 1} \sigma_{Wjt} \sigma_{pjt} \right\} \]  

(A4)

The FOC for aggregate risk exposure gives

\[ \sigma_{Wjt} = \frac{\kappa_t}{\gamma_j} + \frac{1}{\gamma_j} \sigma_{pjt} \]  

(A5)

The FOC for consumption gives

\[ C_{jt} = \frac{1}{p_{jt}} W_{jt} \]  

(A6)
that is, $p_{jt}$ is the wealth / consumption ratio of the household.

Plugging the optimal consumption rate into the HJB, we obtain an expression for the wealth drift in terms of $r_t$ and $\kappa_t$:

$$\mu_{W,jt} = r_t + \delta + \sigma_{W,j} \kappa_t - \frac{1}{p_{jt}}$$

$$= \psi_j (r - \rho) + \frac{1}{2 \gamma_j} \kappa_t^2 + \frac{\psi_j - \psi_j}{\psi_j - 1} \kappa_t \sigma_{p_{jt}} + \frac{1}{2(\psi_j - 1)} \sigma_{p_{jt}}^2 + \mu_{p_{jt}} \tag{A7}$$

**Proof of Proposition 2.** Applying Ito’s lemma, the law of motion of the wealth per capita in group $A$ is:

$$d \left( \int_{t}^{\infty} W_{it}/(\pi_A N_t) \right) = d \left( \int_{-\infty}^{t} (\delta + \eta)e^{-(\delta + \eta)(t-s)} W_{Ais} ds \right)$$

$$= \int_{-\infty}^{t} (\delta + \eta)e^{-(\delta + \eta)(t-s)} W_{Ais} ds + \pi_A N_t (\delta + \eta) W_{Ait} dt$$

$$- (\delta + \eta) dt \int_{-\infty}^{t} (\delta + \eta)e^{-(\delta + \eta)(t-s)} W_{Ais} ds \tag{A8}$$

Dividing by $\int_{t}^{\infty} W_{it}/(\pi_A N_t)$ we obtain

$$\frac{d \left( \int_{t}^{\infty} W_{it}/(\pi_A N_t) \right)}{\int_{t}^{\infty} W_{it}/(\pi_A N_t)} = \frac{dW_{Ais}}{W_{Ais}} + (\delta + \eta) \left( \frac{\pi_A}{x_t} - 1 \right) dt \tag{A9}$$

This gives the law of motion of $x_t$. \hfill \square

**B.2 Human Capital**

For simplicity, the text does not detail what drives $\phi$, the wealth of newborn households relative to the economy. Here, I relate $\phi$ to the discounted value of some labor income flow.

**Labor Income** An agent $i$ born at time $s_i$ is endowed with the labor income process $L_i = \{L_{it} : t \geq s_i\}$, given by

$$L_{it} = G(t - s_i) \times \frac{Y_t}{N_t} \tag{A10}$$

$G(t - s)$, captures the evolution of labor income with age. Without loss of generality, I assume that:

$$\int_{-\infty}^{t} (\delta + \eta)e^{-(\delta + \eta)(t-s)} G(t - s) ds = 1 \tag{A11}$$

Adding labor income received by households of all ages gives $\omega Y_t$. Therefore $\omega$ corresponds to the fraction of the endowment distributed as labor income.
**Budget Constraint** Denote \( \theta_i \) the dollar amount of asset invested in the representative firm. The budget constraint of agent \( i \) is:

\[
A_{is_i} = 0 \\
dA_{it} = (L_{it} - C_{it} + (r_t + \delta)A_{it} + \theta_i(\mu_{Rt} - r_t))dt + \theta_i\sigma_{Rt}dZ_t \text{ for } t \geq s_i
\] (A12)

**Human Capital** Because markets are dynamically complete, a change of variable is useful simplify the problem. Denote \( H_{it} \) the present value at time \( t \) of the labor income of agent \( i \), i.e.

\[
\Lambda_t H_{it} = E_t \left[ \int_t^{+\infty} \Lambda_u e^{-\delta(u-t)} L_{iu} du \right] \tag{A13}
\]

Using the expression for \( L_{iu}x \) and the definition of \( \phi_t \) above, we get

\[
\Lambda_t Y_{it} p_t \phi_t = E_t \left[ \int_{u=t}^{+\infty} \Lambda_u e^{-\delta(u-t)} \omega Y_u G(u-t) \right] \tag{A14}
\]

where \( \Lambda_t \) denotes the stochastic discount factor, which evolves as

\[
\frac{d\Lambda_t}{\Lambda_t} = -r_t dt - \kappa_t dZ_t \tag{A15}
\]

Define \( W_{it} \) the sum of asset and human capital, i.e. \( W_{it} = A_{it} + H_{it} \). Moreover, denote \( p_t \) the total wealth in the economy, divided by total consumption. We can rewrite the agent budget constraint as:

\[
W_{is_i} = \chi_i \phi_{s_i} p_{s_i} Y_{s_i} \\
dW_{it} = ((r_t + \delta)W_{it} + \kappa_t \sigma_{W_{it}} - C_{it}) dt + \sigma_{W_{it}} dZ_t \text{ for } t \geq s_i
\] (A16)

which corresponds to the form in the main text.

**B.3 Solving the Model**

Assume that \( G(u) \) is a sum of \( K \) exponential

\[
G_k(u) = B_k e^{-\delta_k u} \forall 1 \leq k \leq K \\
G(u) = \sum_{1 \leq k \leq K} G_k(u) \tag{A17}
\]

where the coefficients \( (B_k)_{1 \leq k \leq K} \) are such that total aggregate earnings equal \( \omega Y_t \)

\[
1 = \sum_{1 \leq k \leq K} B_k \frac{\delta + \eta}{\delta + \eta + \delta_k} \tag{A18}
\]

Demote \( p_k' \) as the price-dividend of a claim with exponentially decreasing endowment at rate \( \delta + \delta_k \), for \( 1 \leq k \leq K \). Conjecture that this process follows a diffusion process

\[
\frac{dp_k'}{p_k'} = \mu_{p_k'} dt + \sigma_{p_k'} dZ_t
\] (A20)
Using Equation (A13), we obtain
\[
\phi = \omega \sum_{1 \leq k \leq K} B_k \frac{p_k^l}{p} \tag{A21}
\]
We also note that
\[
\frac{1}{p} = \frac{x}{p_A} + \frac{1-x}{p_B} \tag{A22}
\]
gives the first and second derivative of \( p \) with respect to \( x \) in terms of the first and second derivatives of \( p_A \) and \( p_B \) with respect to \( x \).

**Solve for \( \sigma_x \)** Applying Ito’s lemma, we have
\[
\sigma_{p_j} = \frac{\partial x p_j}{p_j} \sigma_x \text{ for } j \in \{ A, B \} \tag{A23}
\]
\[
\sigma_p = \frac{\partial x p}{p} \sigma_x \tag{A24}
\]
\[
\sigma_{p_k^l} = \frac{\partial x p_k^l}{p_k^l} \sigma_x \text{ for } l \in 1 \leq k \leq K \tag{A25}
\]
Substituting the expression for \( \kappa \) in Equation (21) in Proposition 2, we can solve for \( \sigma_x \):
\[
\sigma_x = \frac{x(1-x)\Gamma(\gamma_B - \gamma_A)\sigma}{1 - x(1-x)\Gamma(\gamma_B - \gamma_A)\sigma + \frac{1-\gamma_A}{\psi_A-1} x\sigma_{p_y} p_A - \frac{1-\gamma_B}{\psi_B-1} x\sigma_{p_y} p_B} \tag{A26}
\]

**Solve for \( \mu_x \)** The growth of \( x \) can be expressed in terms of previously computed quantities:
\[
\mu_x = x(1-x) \left( \kappa (\sigma_{W_j} - \sigma_{W_j}) - \frac{1}{p_A} + \frac{1}{p_B} \right) + (\delta + \eta)\phi(x - \pi_A) + x\sigma_{p_y} p + x\sigma_{p_y} p_B \tag{A27}
\]
Using Ito’s lemma, we can express the drift of all quantities to solve for:
\[
\mu_{p_j} = \frac{\partial x p_j}{p_j} \mu_x + \frac{1}{2} \frac{\partial x p_j}{p_j} \sigma_x^2 \text{ for } j \in \{ A, B \} \tag{A28}
\]
\[
\mu_p = \frac{\partial x p}{p} \mu_x + \frac{1}{2} \frac{\partial x p}{p} \sigma_x^2 \tag{A29}
\]
\[
\mu_{p_k^l} = \frac{\partial x p_k^l}{p_k^l} \mu_x + \frac{1}{2} \frac{\partial x p_k^l}{p_k^l} \sigma_x^2 \tag{A30}
\]

**System of ODEs** After obtaining the interest rate with Equation (25), the budget constraint for \( A, B \) and the definition for \( (p_k^l)_{1 \leq k \leq K} \) gives a system of \( 2 + K \) ODEs:
\[
\frac{1}{p_j} + \mu_{W_j} = r + \delta + \kappa \sigma_{W_j} \text{ for } j \in \{ A, B \} \tag{A31}
\]
\[
\frac{1}{p_k^l} + \mu - \delta_k + \sigma_{p_k^l} = r + \delta + \kappa (\sigma + \sigma_{p_k^l}) \text{ for } 1 \leq k \leq K \tag{A32}
\]
where \( \mu_{W_j} \) and \( \sigma_{W_j} \) are given by Proposition 1.
B.4 Computational Method

The system of PDEs is written on a state space grid and derivatives are substituted by finite difference approximations. Importantly, first order derivatives are upwinded. This allows to naturally handle boundary conditions at the frontiers of the state space.

Denote $Y$ the solution and denote $F(Y)$ the finite difference scheme corresponding to a model. The goal is to find $Y$ such that $F(Y) = 0$. I solve for $Y$ using a fully implicit Euler method. Updates take the form

$$\forall t \leq T \quad 0 = F(y_{t+1}) - \frac{1}{\Delta}(y_{t+1} - y_t) \quad (A33)$$

Each time step requires to solve a non-linear equation. I solve this non-linear equation using a Newton-Raphson method. The Newton-Raphson method converges if the initial guess is close enough to the solution. Since $y_t$ converges towards $y_{t+1}$ as $\Delta$ tends to zero, one can always choose $\Delta$ low enough so that the inner steps converge. Therefore, I adjust $\Delta$ as follows. If the inner iteration does not converge, I decrease $\Delta$. If the inner iteration converges, I increase $\Delta$. After a few successful implicit time steps, $\Delta$ is large and, therefore, the algorithm becomes like Newton-Raphson. In particular, the convergence is quadratic around the solution.

This method is most similar to a method used in the fluid dynamics literature, called the Pseudo-Transient Continuation method. Formal conditions for the convergence of the algorithm are given in Kelley and Keyes (1998). The algorithm with $I = 1$ and $\Delta$ constant corresponds to Achdou et al. (2016). Allowing $I > 1$ and adjusting $\Delta$ are important to ensure convergence in the case on non-linear PDEs.

Appendix C

Lemma 1 (Kolmogorov-Forward with Aggregate Risk). Suppose $w_t$ is a process evolving according to

$$\frac{dw_t}{w_t} = \mu_t dt + \sigma_t dZ_t \quad (A34)$$

where $Z_t$ is a standard aggregate Brownian Motion. Suppose that agents die with Poisson rate $\delta$ and, are born according to the density $g_{\chi t}$. Finally suppose that population grows with rate $\eta$. The density of $w_t$, $g_t$, follows the law of motion:

$$dg_t = \left( -\mu_t \partial_w(wg_t) + \frac{1}{2} \sigma_t^2 \partial_w^2(w^2 g_t) + (\delta + \eta)(g_{\chi t} - g_t) \right) dt - \sigma_t \partial_w(wg_t) dZ_t \quad (A35)$$

Proof of Lemma 1. For any function $f$, we have

$$\int_{-\infty}^{+\infty} f(w) g_{t+dt}(w) dw = \int_{-\infty}^{+\infty} (f(w) + df(w)) g_t(w) + f(w)(\delta + \eta)dt(g_{\chi t}(w) - g_t(w)) ) dw \quad (A36)$$

Assuming that $f$ is twice differentiable, Ito’s lemma gives:

$$\int_{-\infty}^{+\infty} f(w) dg_t(w) dx = \int_{-\infty}^{+\infty} \left( \mu_t wdt \partial_w f(w) + \frac{1}{2} \sigma_t^2 w^2 dt \partial_w^2 f(w) + \sigma_t \partial_w f(w) dZ_t \right) g_t(w) dw$$

$$+ \int_{-\infty}^{+\infty} f(w)(\delta + \eta)dt(g_{\chi t}(w) - g_t(w) ) dw \quad (A37)$$
Assume that $f$ decays fast enough as $|x| \to +\infty$ and use integration by parts to obtain
\[
\int_{-\infty}^{+\infty} f(w)dg_t(w)dw = \int_{-\infty}^{+\infty} f(w) \left( -\mu_t \partial_w(wg_t) + \frac{1}{2} \sigma_t^2 \partial_{ww}(w^2g_t) + (\delta + \eta)dt(g_{\chi_t} - g_t) dw \right) dt dw
- \int_{-\infty}^{+\infty} f(w)\sigma_t \partial_w(wg_t) dZ_t dw
\]
This equality must hold for all $f$ satisfying the conditions above. Therefore, we obtain
\[
dg_t = \left( -\mu_t \partial_w(wg_t) + \frac{1}{2} (\sigma_t^2 + \nu_t^2) \partial_{ww}(w^2g_t) + (\delta + \eta)(g_{\chi_t} - g_t) \right) dt - \sigma_t \partial_w(wg_t) dZ_t
\]  
(A39)

Proof of Proposition 4. The proposition follows from Lemma 1 applied to the particular model. \[\square\]

**Lemma 2** (Stability of Linear Functional). Let $x_t \in \mathbb{R}$ a continuous-time strong Markov process non explosive, irreducible, positive recurrent, with unique invariant probability measure.

The process
\[
dM_t = (\mu(x_t)M_t + b(x_t)) dt + \sigma(x_t)M_t dW_t
\]  
(A40)

with $P(b(x) \geq 0) = 1$ and $P(b(x) > 0) > 0$ converges to infinity a.s. iff $\mathbb{E}[\mu(x) - \frac{1}{2} \sigma(x)^2] > 0$ where $\mathbb{E}$ denotes the expectation with respect to the invariant probability measure of $x$.

Proof of Lemma 2. The proof for the case $b(x_t) = 0$ (purely multiplicative process) is in Maruyama and Tanaka (1959). I extend the theorem to the case $b(x_t) \neq 0$.

The process $M_t$ that can be written:
\[
M_t = \int_0^t e^{\int_s^t \left( (\mu(x_u) - \frac{1}{2} \sigma(x_u)^2) du + \sigma(x_u) dW_u \right) } b(x_u) ds + M_0
\]
In particular, it satisfies the following recurrence equation:
\[
M_{t+\tau} = e^{\int_0^{t+\tau} \left( (\mu(x_u) - \frac{1}{2} \sigma(x_u)^2) du + \sigma(x_u) dW_u \right) } M_t + \int_t^{t+\tau} e^{\int_s^{t+\tau} \left( (\mu(x_u) - \frac{1}{2} \sigma(x_u)^2) du + \sigma(x_u) dW_u \right) } b(x_u) ds
\]
Denote $I$ the set of values that $x_1$ can take. Take $a < b$, both in $I$. Define the sequence of stopping times $S_0 = 0$ and
\[
T_n \equiv \inf\{t > S_n; x_t = a\}
S_n \equiv \inf\{t > T_n; x_t = b\}
\]
Define
\[
Y_n = M_{T_n}
A_n = \exp \left( \int_{T_n}^{T_{n+1}} ((\mu(x_u) - \frac{1}{2} \sigma(x_u)^2) du + \sigma(x_u) dW_u) \right)
B_n = \int_{T_n}^{T_{n+1}} e^{\int_{T_n}^{T_{n+1}} ((\mu(x_u) - \frac{1}{2} \sigma(x_u)^2) du + \sigma(x_u) dW_u) } b(x_u) ds
\]  
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The sequence $Y_n$ satisfies the following recurrence relation:

$$Y_{n+1} = A_n Y_n + B_n$$

where $A_n$ and $B_n$ are i.i.d over time. Moreover, $A_1$ is positive a.s., $B_1$ is non negative a.s. with $P(B_1 = 0) < 1$ and $E[\log(B_1)] < +\infty$, and $P(A_1 x + B = x) < 1$ for any $x \in \mathbb{R}$. As proven by Goldie and Maller (2000), $Y_n$ converges a.s. to infinity a.s. iff $E[\log A_1] \geq 0$ i.e. $E \left[ \int_{T_1}^{T_2} (\mu(x_u) - \frac{1}{2} \sigma(x_u)^2) \, du \right] \geq 0$. Finally, Maruyama and Tanaka (1959) show that, for any integrable function $f$, $E \left[ \int_{T_1}^{T_2} f(x_u) \, du \right] \geq 0$ iff $E[f(x)] \geq 0$. I conclude by noting that $M_t$ converges to infinity a.s. if and only if $Y_n$ converges to infinity a.s. because, for $t \in (T_n, T_{n+1}]$, $Y_n \leq M_t \leq Y_{n+1}$.

\[ \square \]

**Lemma 3 (Tail Exponent).** Let $x_t \in \mathbb{R}$ a continuous-time strong Markov process non explosive, irreducible, positive recurrent, with unique invariant probability measure.

Suppose the dynamics of individual wealth $w_{it}$ has the following law of motion:

$$\frac{dw_{it}}{w_{it}} = \mu(x_t) dt + \sigma(x_t) dZ_t + \nu(x_t) dW_{it}$$  \hspace{1cm} (A41)

where $W_{it}$ is an idiosyncratic Brownian Motion and $Z_t$ is an aggregate Brownian Motion. Moreover, assume that individuals die with death rate $\delta > 0$ and are re-injected according to a distribution $g_{xt}$ with thin tails.\(^{35}\)

The wealth distribution is fat-tailed with tail index $\zeta$ if and only if there is $\zeta > 0$ such that:

$$\zeta E[\mu(x)] - \frac{1}{2} E[\sigma(x)^2] + \frac{\zeta(\zeta - 1)}{2} E[\nu(x)^2] - \delta = 0$$  \hspace{1cm} (A42)

where $E$ denotes the expectation with respect to the stationary density of $x$.

**Proof of Lemma 3.** Denote $m_{it}^\xi$ the $\xi$-moment of the wealth distribution and $m_{xt}^\xi$ the $\xi$ cross-sectional moment of human capital. The law of motion of $m_{it}^\xi$ is given by:

$$dm_{it}^\xi = \left( \xi \mu(x_t) + \frac{\xi(\xi - 1)}{2} (\sigma(x_t)^2 + \nu(x_t)^2) m_{it}^\xi + (\delta + \eta)(m_{it}^\xi - m_{it}^\xi) \right) dt + \xi \sigma(x_t) m_{it}^\xi dZ_t$$  \hspace{1cm} (A43)

Compared to the dynamics of the wealth density, the dynamics off $m_{it}^\xi$ can be examined in isolation for each $\xi$, since it does not depend on the derivative of the function $\xi \to m_{it}^\xi$. This insight is due to Gabaix et al. (2016) and Luttmer (2012), that show that, in setups where wealth dynamics are linear in wealth, it is often easier to work with higher-order moments of wealth. My contribution is to study the dynamics of these higher-order moments in a Markov economy.

Denote $f$ the function defined as:

$$f(\xi) = \xi \left( E[\mu(x)] - \frac{1}{2} E[\sigma(x)^2] \right) + \frac{\xi(\xi - 1)}{2} E[\nu(x)^2] - (\delta + \eta)$$  \hspace{1cm} (A44)

\(^{35}\)Formally, the $\xi$th moments of $\psi$ exists for $\xi \leq \zeta$.

\(^{36}\)It is enough that $E[\nu(x)] > 0$ or $E[\mu(x) - \frac{1}{2} \sigma(x)^2] > 0$. 

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Lemma 2 says that \( m_0^\xi \) converges to infinity a.s. iff \( f(\xi) = 0 \). Since there exists \( \zeta > 0 \), such that \( f(\xi) = 0 \), it means that, for \( \xi > 0 \), \( f(\xi) \) is positive if and only if \( \xi \in [\zeta, +\infty) \). Indeed, if \( \mathbb{E}[\nu(x)^2] \neq 0 \), \( f(\xi) \) is a quadratic equation, and the product of its roots is negative. Otherwise, \( f(\xi) \) is linear in \( \xi \). In both cases, the \( \xi \) moment converge to infinity a.s. if and only if \( \xi \geq \zeta \).

**Proof of Proposition 5.** The proposition follows from Lemma 3 applied to the particular model.

**Proof of Proposition 6.** Using Proposition 2, we get

\[
d\ln x_t = \left( \mu_{w,At} - \frac{1}{2} \sigma_{w,At}^2 + (\delta + \eta) \left( \frac{\pi_A}{x_t} \phi_t - 1 \right) \right) dt + \sigma_{w,At} dZ_t \tag{A45}
\]

Since \( \ln(x_t) \) is ergodic, its drift must average to zero. This gives:

\[
0 = \mathbb{E} \left[ \mu_{w,At} - \frac{1}{2} \sigma_{w,At}^2 \right] + (\delta + \eta) \left( \mathbb{E} \left[ \frac{\pi_A}{x_t} \phi_t \right] - 1 \right) \tag{A46}
\]

This expression allows me to replace \( \mathbb{E} \left[ \mu_{w,At} - \frac{1}{2} \sigma_{w,At}^2 \right] \) in the expression for the tail index.

**Proof of Proposition 7.** This proof adapts Gomez (2018) to the case of this model. Applying Ito’s lemma on Equation (45) gives the law of motion of the quantile \( q_t \)

\[
0 = -g_t(q_t) \frac{dq_t}{dt} + \int_{q_t}^{+\infty} \frac{dg_t(w)}{dt} dw - \sigma[q_t][\sigma[dq_t]] \tag{A47}
\]

where \( \sigma[dg_t(q_t)] \) and \( \sigma[dq_t] \) denote respectively the volatility of \( g_t(q_t) \) and \( q_t \). Applying Ito’s lemma on Equation (46) gives the law of motion of the top share \( S_t \):

\[
dS_t = -q_t g_t(q_t) dq_t + \int_{q_t}^{+\infty} ndg_t(w) dw - q_t \sigma[dg_t(q_t)] \sigma[dq_t] dt - \frac{1}{2} g_t(q_t) \sigma[dq_t]^2 dt \tag{A48}
\]

Using the law of motion for \( q_t \) from Equation (A47), we obtain the law of motion of \( S_t \):

\[
dS_t = \int_{q_t}^{+\infty} (w - q_t) dg_t(w) dw - \frac{1}{2} g_t(q_t) \sigma[dq_t]^2 dt
= \int_{q_t}^{+\infty} (w - q_t) dg_t(w) dw - \frac{1}{2} g_t(q_t) \left( \int_{q_t}^{+\infty} \sigma[dg_t(w)] dw \right)^2 dt \tag{A49}
\]

where

\[
dg_t = \pi_A dg_{At} + (1 - \pi_A) dg_{Bt} \tag{A50}
\]

and the law of motion of \( g_{jt} \) for \( j \in \{A, B\} \) is given by the Kolmogorov-Forward equation from Lemma 3

\[
dg_{jt} = \left( -\mu_{wjt} \partial_w(wg_{jt}) + \frac{1}{2} \sigma_{wjt}^2 \partial_{ww}(w^2 g_{jt}) + (\delta + \eta)(g_{At} - g_{jt}) \right) dt - \sigma_{wjt} \partial_w(wg_{jt}) dZ_t \tag{A51}
\]
Plugging it into (A49), we obtain

\[ dS_t = \int_{q_t}^{\infty} (w - q_t) \left( \sum_{j \in \{A, B\}} \pi_j \left( \left( -\mu_{w, j} \partial_w g_{jt}(w) + \frac{1}{2} \sigma_{w, j}^2 \partial_{ww} g_{jt}(w) \right) dt - \sigma_{w, j} \partial_w g_{jt}(w) dZ_t \right) \right) dw 
+ \int_{q_t}^{\infty} (w - q_t) \left( \sum_{j \in \{A, B\}} \pi_j (\delta + \eta) (g_{xt} - g_{jt}(w)) \right) dt 
- \frac{1}{2g_t(q_t)} \left( \int_{q_t}^{\infty} \sum_{j \in \{A, B\}} \pi_j \sigma_{w, j} \partial_w (wg_{jt}(w)) \right)^2 dt dw \]  

(A52)

Integrating by parts and rearranging:

\[ dS_t = \left( \int_{q_t}^{\infty} \sum_{j \in \{A, B\}} \mu_{w, j} \pi_j g_{jt}(w) \right) dt + \left( \int_{q_t}^{\infty} \sum_{j \in \{A, B\}} \sigma_{w, j} \pi_j g_{jt}(w) dw \right) dZ_t 
+ (\delta + \eta) dt \left( q_t \alpha - S_t + \int_{q_t}^{\infty} (w - q_t) g_{xt}(w) dw \right) 
+ \frac{g_t^2(q_t)}{2} \left( \int_{q_t}^{\infty} \sum_{j \in \{A, B\}} \sigma_{w, j}^2 \pi_j g_{jt}(w) g_t(q_t) - \left( \sum_{j \in \{A, B\}} \sigma_{w, j} \pi_j g_{jt}(q_t) \right)^2 \right) \]  

(A53)

Dividing by \( S_t \), we obtain Proposition 7. \( \square \)

Appendix D

D.1 Proofs

Proof of Proposition 3. I derive a Campbell-Shiller decomposition in continuous-time. The cumulative return of a dollar invested in the asset follows the process:

\[ \frac{dR_t}{R_t} = \frac{1}{p_{Dt}} + \frac{d(p_{Dt} D_t)}{p_{Dt} D_t} \]  

(A54)

Therefore, in log,

\[ d \log R_t = \frac{1}{p_{Dt}} dt + d \log p_{Dt} + d \log D_t \]  

(A55)

As in Chacko and Viceira (2005), I log-linearize the dividend-price ratio around the average value of \( \log p_D \):

\[ \frac{1}{p_{Dt}} \approx \alpha (1 - \log \alpha) - \alpha \log p_{Dt} \]  

(A56)

with \( \alpha = e^{-\mathbb{E}[\log p_D]} \). Plugging this expression into Equation (A55), I obtain:

\[ d \log R_t \approx \alpha (1 - \log \alpha) dt - \alpha \log p_{Dt} dt + d \log p_{Dt} + d \log D_t \]  

(A57)
Integrating, the price-dividend ratio can be written as:

$$\log p_{Dt} \approx E_t \left[ \int_t^{+\infty} e^{-\alpha(s-t)}(\alpha(1 - \log \alpha)dt + d\log R_s - d\log D_s) \right]$$

$$\approx 1 - \log \alpha + E_t \left[ \int_t^{+\infty} e^{-\alpha(s-t)}(d\log D_s - d\log R_s) \right]$$

(A58)

In the Markovian economy of Section 5, the expected growth rate of log-dividend is constant. Deriving with respect to the state variable $x$, one obtains:

$$\frac{\partial \log(p_D)}{\partial x} \approx -\int_t^{+\infty} e^{-\alpha(s-t)} \frac{\partial E_t[d\log R_s|x_t = x]}{\partial x}$$

(A59)

D.2 Calibration of Human Capital

I calibrate the human capital using the life cycle evolution of labor income, following Gârleanu and Panageas (2015). The life cycle income of households $G(u)$ is a sum of two exponentials approximating the hump shaped pattern of earnings observed in the data:

$$G(u) = B_1 e^{-\delta_1 u} + B_2 e^{-\delta_2 u}$$

(A60)

with $B_1 = 30.72, B_2 = -30.29$.

I chose the share of endowment distributed as labor income as $\omega = 92\%$, following Gârleanu and Panageas (2015). Exploring a lower value for the labor share would only strengthen my result: since a lower labor share decreases the wealth of arriving agents, it rises the average wealth growth of existing households. In equilibrium, this tends to increase the interest rate. But the model already tends to give an interest rate that is too high, according to Table 5.

D.3 Partial Equilibrium Estimation

I estimate a partial equilibrium version of the model, which allows me to answer the following question: given the observed interest rate and market price of risk, what preference parameters can match moments about the wealth distribution? More precisely, I consider an economy with a constant interest rate equal to 2.7% and constant market price of risk equal to 0.29. I consider agents with homogeneous subjective discount rate $\rho$ and EIS $\psi$, but heterogeneous RRA $\gamma$. I estimate $\rho, \psi, \gamma_A, \gamma_B$ to match aggregate wealth growth, the aggregate wealth exposure to stock market return, the exposure of top wealth shares to stock market returns, and the tail index of the wealth distribution. Table A3 shows that a model with a relatively small degree of heterogeneity in risk aversion can easily generate the wealth moments documented in the paper. In other words, given the large compensation for risk observed in the data, a small heterogeneity in risk aversion is enough to generate a right tail of the wealth distribution as thick as the data.
Table A1: The Equity Share Increases Across the Wealth Distribution

<table>
<thead>
<tr>
<th>Groups of Households Defined by Wealth Percentiles</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td>All Households</td>
</tr>
<tr>
<td>Equity Share</td>
</tr>
<tr>
<td>Public Equity</td>
</tr>
<tr>
<td>Private Equity</td>
</tr>
<tr>
<td>Non Actively Managed</td>
</tr>
<tr>
<td>Actively Managed</td>
</tr>
</tbody>
</table>

**Panel A: All Sample**

| Equity Share among Stockholders       | 44.7%     | 56.0%        | 65.9%     | 76.0%     |

**Panel B: Stockholders**

| Is Stockholder             | 45.9%     | 90.7%        | 91.2%     | 91.0%     |
| Equity Share among Stockholders | 44.7%   | 56.0%        | 65.9%     | 76.0%     |

**Panel C: Entrepreneurs**

| Is Entrepreneur           | 10.5%     | 62.1%        | 69.8%     | 78.5%     |
| Equity Share among non-Entrepreneurs | 26.8% | 40.7%        | 50.0%     | 57.9%     |

**Panel D: Stock Options Holders**

| Received Stock Options     | 6.4%      | 11.2%        | 11.5%     | 6.1%      |
| Equity Share among non Stock Options Holders | 44.7% | 56.0%        | 65.9%     | 76.0%     |

| Share of Total Wealth         | 20.7%     | 7.7%         | 3.8%      |
| Labor Income / Wealth         | 12.6%     | 2.9%         | 1.6%      | 0.7%      |

**Notes:** Data from SCF 1989-2013. The variable Equity Share is defined as private equity + public equity over networth: \( \frac{(\text{equity} + \text{bus})}{\text{networth}} \). Stockholders are defined as the households that hold public equity. Entrepreneurs are defined as the households with an active management role in one of the company they invest in.
Table A2: The Exposure to Stock Returns Across the Wealth Distribution:  
**Saez and Zucman (2016)** Series

<table>
<thead>
<tr>
<th>Wealth Growth Within Percentile Thresholds</th>
<th>Saez and Zucman (2016)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 – 0.1%</td>
</tr>
<tr>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>Excess Stock Returns</td>
<td>0.49***</td>
</tr>
<tr>
<td></td>
<td>(0.10)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.19</td>
</tr>
<tr>
<td>$N$</td>
<td>96</td>
</tr>
</tbody>
</table>

*Notes: The table reports the results of the regression of the growth of the wealth growth of households in a given percentile group on asset returns. Estimation is via OLS. Standard errors in parentheses and estimated using Newey-West with 3 lags. *,**, *** indicate significance at the 0.1, 0.05, 0.01 levels, respectively.*

Table A3: Partial Equilibrium Estimation

<table>
<thead>
<tr>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
</tr>
<tr>
<td>Fixed Parameters</td>
<td>Population share $\pi_A$</td>
</tr>
<tr>
<td>Estimated Parameters</td>
<td>RRA $\gamma_A$</td>
</tr>
<tr>
<td></td>
<td>RRA $\gamma_B$</td>
</tr>
<tr>
<td></td>
<td>Subjective Discount Rate $\psi$</td>
</tr>
<tr>
<td></td>
<td>Subjective Discount Rate $\rho$</td>
</tr>
<tr>
<td>Total Wealth</td>
<td>$\beta_{R Mt \to \Delta \ln w_t}$</td>
</tr>
<tr>
<td></td>
<td>Average Growth</td>
</tr>
<tr>
<td>Top Wealth</td>
<td>$\beta_{R Mt \to \Delta \ln s_i}$</td>
</tr>
<tr>
<td></td>
<td>Tail Index $\zeta$</td>
</tr>
<tr>
<td>J-Statistic</td>
<td>0.00</td>
</tr>
</tbody>
</table>

*Notes: Estimation of the model in partial equilibrium.*
Figure A1: The Equity Share Increases Across the Wealth Distribution

(a) Top Percentiles Linearly Spaced  
(b) Top Percentiles Log-linearly Spaced

Notes. Figure A1a plots the average equity share within 20 linearly spaced percentile bins in the wealth distribution. Figure A1b plots the average equity share within 20 logarithmically spaced percentile bins in the wealth distribution. The horizontal line represents the average equity share. The vertical line splits the set of households in two: households on either side of the vertical line own half of total wealth (this corresponds to top percentile ≈ 3%). Data from the Survey of Consumer Finance (SCF), a cross-sectional survey of US households from 1989 to 2013. The equity share is constructed as \((\text{equity} + \text{bus}) / \text{networth}\).